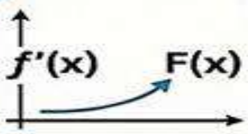
أوراق عمل كامل الوحدة الخامسة التكامل

Unit 5: Integration

وحدة 5: التكامل

الجزء الأول: أساسيات ومفاهيم التكامل

الدرس 1: الدالة الأصلية (عكس المشتقة) |
| Lesson 1: Antiderivative



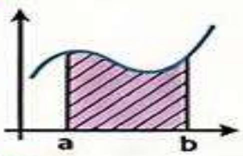
فهم المفهوم الأساسي للتكامل كعملية عكسية لعملية الاشتقاق.

الدرس 2: المجموع ورمز سيجما |
| Lesson 2: Sums and Sigma Notation

$$\sum_{i=1}^{n}$$

استخدام الرموز الرياضية للتعبير عن المجاميع المتسلسلة.

الدرس 3: المساحة تحت منحنى |
| Lesson 3: Area Under a Curve



الربط بين مفهوم التكامل وحساب المساحات الهندسية المعقدة.

الجزء الثاني: النظريات والتطبيقات

الدرس 4: التكامل المحدود |
| Lesson 4: The Definite Integral

$$\int_a^b f(x)\,dx$$

حساب قيمة التكامل ضمن فترات محددة للحصول على نتائج عددية.

الدرس 5: النظرية الأساسية في التفاضل والتكامل
| Lesson 5: The Fundamental Theorem of Calculus

$$\frac{d}{dx} \quad \int_a^b F(b) - F(a)$$

الربط الجوهري بين فرعي التفاضل والتكامل والتكامل في نظرية واحدة.

الدرس 6: التكامل بالتعويض |
| Lesson 6: Integration by Substitution

$$\int_a f(g(x))g'(x)dx \;\downarrow\; u = g(x) \;\rightarrow\; \int_u^u$$

تعلم تقنية قوية لتبسيط وحل دوال التكامل المعقدة.

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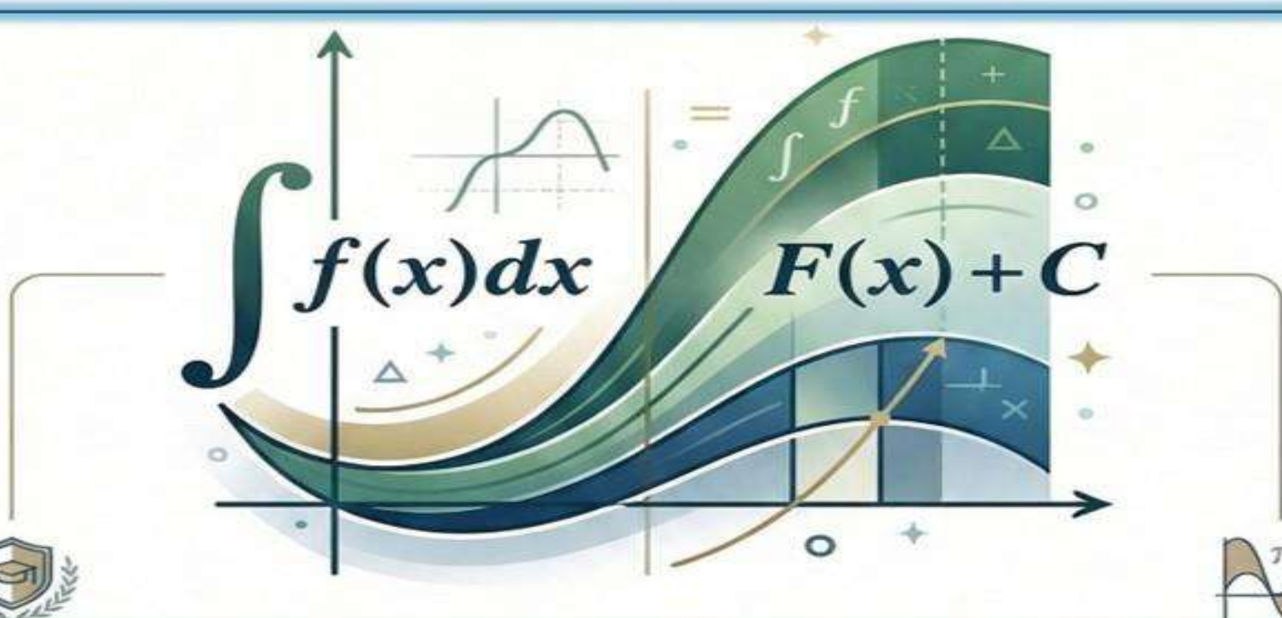
Unit 5: Integration

وحدة 5: التكامل



Lesson 1 : Antiderivative

الدرس الأول : عكس المشتقة (الدالة الأصلية)

Grade 12 Advanced

الثاني عشر متقدم

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0505114830

Student Name (اسم الطالب:) ______________________________

Class (الصف) ______________________________

Unit 5

Integration

Lesson 1

Antiderivatives

Objectives

- Find the antiderivative of a given function.
- Understand the notion of indefinite integral as finding an antiderivative.
- Compute straightforward indefinite integrals.

Example: 1 Find an antiderivative of $f(x) = x^2$

Solution

THEOREM 1.1

Suppose that F and G are both antiderivatives of f on an interval I. Then.
$G(x) = F(x) + c$, For some constant c.

COROLLARY 10.1

Suppose that $g'(x) = f'(x)$ for all x in some open interval I. Then, for some constant c,

$$g(x) = f(x) + c, \text{ for all } x \in I.$$

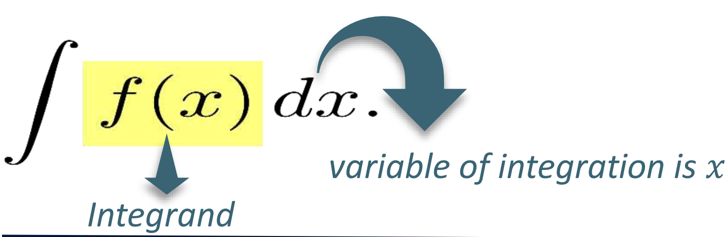
DEFINITION 1.1

Let F be any antiderivatives of f on an interval I. The **Indefinite Integral** of $f(x)$ (*with respect to* x) on I, is defined by.

$$\int f(x)dx = F(x) + c,$$

Where c is an *arbitrary constant* (the **constant of integration**).

Integration: *The process of computing the integral*



Integration as Inverse of Differentiation

$$\frac{d}{dx}[x^r] = rx^{r-1}$$

$$\int x^r dx = \frac{x^{r+1}}{r+1} + c$$

$$\frac{d}{dx}\left(\int f(x)\,dx\right) = f(x)$$

$$\int \frac{d}{dx}(f(x))\,dx \text{ or } \int f'(x)\,dx = f(x) + c$$

THEOREM 1.2 (Power Rule)

For any power $r \neq -1$,

$$\int x^r dx = \frac{x^{r+1}}{r+1} + c,$$

Here, if $r < -1$, the interval I on which this is defined can be any interval that does not include $x = 0$.

Example: 1 Evaluate

1) $\int x^{17} dx$

2) $\int x^3 dx$

3) $\int x\,dx$

4) $\int dx$

5) $\int m^4 dm$

6) $\int \frac{1}{x^3}\,dx$

Exercise: 1.1 Evaluate

1) $\int x^{-5} dx$

2) $\int \sqrt{x}\,dx$

3) $\int \frac{1}{\sqrt[3]{x}}\,dx$

Derivative	Antiderivative=Integration
$\frac{d}{dx}(F(x)) = f(x)$	$\int f(x)\,dx = F(x) + c$
$\frac{d}{dx}(a) = 0$ $\quad \frac{d}{dx}(ax) = a$	$\int a\,dx = ax + c$
$\frac{d}{dx}(x^2) = 2x$	$\int x\,dx = \frac{x^2}{2} + c$
$\frac{d}{dx}(x^{n+1}) = (n+1)x^n$	$\int x^n\,dx = \frac{x^{n+1}}{n+1} + c \;, n \neq 1$
$\frac{d}{dx}(e^x) = e^x$	$\int e^x\,dx = e^x + c$
$\frac{d}{dx}(e^{-x}) = -e^{-x}$	$\int e^{-x}\,dx = -e^{-x} + c$
$\frac{d}{dx}(a^x) = a^x.\ln a$	$\int a^x\,dx = \frac{1}{\ln a}a^x + c$
$\frac{d}{dx}(\ln x) = \frac{1}{x} = x^{-1}$	$\int x^{-1}\,dx = \int \frac{1}{x}\,dx = \ln x + c$
$\frac{d}{dx}(\sin x) = \cos x$	$\int \cos x\,dx = \sin x + c$
$\frac{d}{dx}(\cos x) = -\sin x$	$\int \sin x\,dx = -\cos x + c$
$\frac{d}{dx}(\tan x) = \sec^2 x$	$\int \sec^2 x\,dx = \tan x + c$
$\frac{d}{dx}(\cot x) = -\csc^2 x$	$\int \csc^2 x\,dx = -\cot x + c$
$\frac{d}{dx}(\sec x) = \sec x\tan x$	$\int \sec x\tan x\,dx = \sec x + c$
$\frac{d}{dx}(\csc x) = -\csc x\cot x$	$\int \csc x\cot x\,dx = -\csc x + c$
$\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$	$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}x + c$
$\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$	$\int \frac{1}{1+x^2}\,dx = \tan^{-1}x + c$
$\frac{d}{dx}(\sec^{-1}x) = \frac{1}{\lvert x\rvert\sqrt{x^2-1}}$	$\int \frac{1}{\lvert x\rvert\sqrt{x^2-1}}\,dx = \sec^{-1}x + c$

THEOREM 1.3

Suppose that $f(x)$ and $g(x)$ have antiderivatives. Then, for any constants, a and b,

$$\int [af(x) + bg(x)]dx = a\int f(x)dx + b\int g(x)dx.$$

Example: 2 Evaluate

1) $\int (x^3 + 2x - 7)dx$

2) $\int (mx + a)\, dx$

3) $\int (3\cos x + 4x^8)dx$

4) $\int \left(2x^{-2} + \frac{1}{\sqrt{x}}\right) dx$

Exercise 2.1 Evaluate

1) $\int \left(3e^x - \frac{2}{1+x^2}\right) dx$

2) $\int \left(3\sqrt{x} - \frac{1}{x^4}\right) dx$

3) $\int (4x - 2e^x)dx$

4) $\int (3cosx - sinx)dx$

5 $\int 2\, secx\, tanx\, dx$

6) $\int \frac{4}{\sqrt{1-x^2}}\, dx$

Example: 3 Find the general antiderivative.

1) $\int \frac{x^{1/3}-3}{x^{2/3}}dx$

2) $\int \frac{x+2x^{3/4}}{x^{5/4}}dx$

3) $\int x^{1/4}(x^{5/4}-4)\,dx$

4) $\int x^{2/3}(x^{-4/3}-3)\,dx$

Exercise3.1

1) $\int 4\frac{cosx}{sin^2x}dx$

2) $\int \frac{e^x+3}{e^x}dx$

3) $\int \frac{3}{4x^2+4}dx$

4) $\int \left(2cosx-\sqrt{e^{2x}}\right)dx$

Derivative of Natural Logarithm Function

THEOREM 1.4

For $x \neq 0$,

$$\frac{d}{dx}\ln|x| = \frac{1}{x}.$$

Using Chain Rule and Theorem 1.4 we get:

$$\frac{d}{dx}[ln|f(x)|] = \frac{\frac{d}{dx}[f(x)]}{f(x)} = \frac{f'(x)}{f(x)}$$ Where $f(x) \neq 0$

Example: 4

For any x for which $tan\ x \neq 0$, Evaluate.

1) $\frac{d}{dx}[ln|tan\ x|]$

Exercise4.1

Find the derivative.

1) $\frac{d}{dx}[ln|sec\ x + tan\ x|]$

2) $\frac{d}{dx}[ln|sin\ x - 2|]$

From theorem 1.4 (differentiation rule)

$$\frac{d}{dx}\ln|x| = \frac{1}{x}, \text{For } x \neq 0$$

Corollary 1.1

In any interval not containing 0,

$$\int \frac{1}{x}dx = \ln|x| + c.$$

Example: 5

Evaluate

1) $\int \left(\frac{3}{x} - 2\cos x\right) dx$

2) $\int \left(\frac{3}{x^2+1} + 7x^{-1}\right) dx$

Corollary 1.2

$$\int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + c.$$

In any interval in which $f(x) \neq 0$.

Example: 6

Evaluate

1) $\int \frac{\sec^2 x}{\tan x}\, dx$

2) $\int \frac{e^x}{e^x+3}\, dx$

Exercise 6.1

1) $\int \frac{\cos x}{\sin x}\, dx$

2) $\int \frac{4x}{x^2+4}\, dx$

3) $\int \tan x\, dx$

4) $\int \cot x\, dx$

Identifying Integrals That We Cannot Yet Evaluate

Example: 7

Which of the integrals below can be evaluated given the rules developed in this section?

1) $\int \frac{1}{\sqrt[3]{x^2}} dx$

2) $\int \sec x\, dx$

3) $\int \frac{2x}{x^2+1} dx$

Exercise 7.1

Which of the integrals below can be evaluated given the rules developed in this section?

1) $\int \frac{x^3+1}{x} dx$

2) $\int (x+1)(x-1)\, dx$

3) $\int x sin2x\, dx$

4) $\int \sqrt{x^3+4}\, dx$

5) $\int \left(\sqrt{x^3}+4\right) dx$

6) $\int \left(\frac{1}{x^2}-1\right) dx$

7) $\int \frac{1}{x^2-1} dx$

8) $\int 2x\, cosx^2\, dx$

9) $\int (x sin2x + x^2 cos2x) dx$

Example: 8

Find the function $f(x)$ satisfying the given conditions:

1) $f'(x) = 3e^{x} + x,\ \ f(0) = 4$

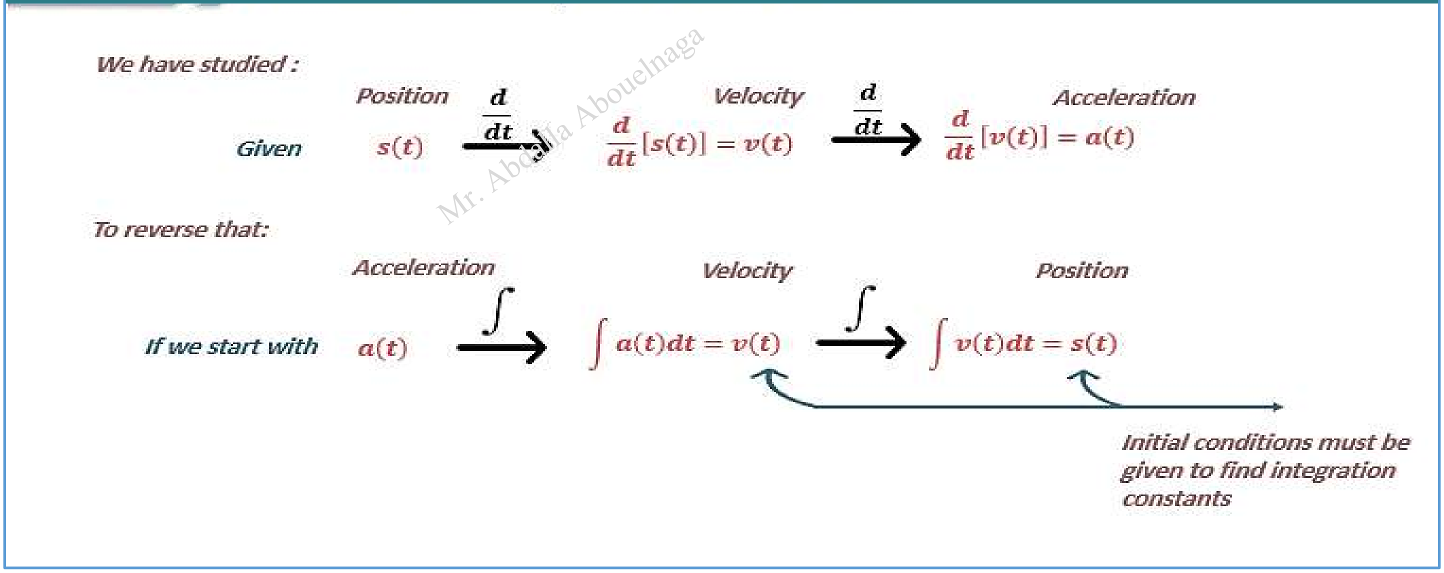
2) $f'(x) = 4cosx,\ \ f(0) = 3$

Exercise 8

1) Find a function $f(x)$ such that the point $(-1,1)$ is on the graph of $y = f(x)$, the slope of the tangent line at $(-1,1)$ is 2 and $f''(x) = 6x + 4$.

2) Find a function $f(x)$ such that the point $(-1,1)$ is on the graph of $y = f(x)$, the slope of the tangent line at $(-1,1)$ is 2 and $f''(x) = 6x + 4$.

Position, Velocity and Acceleration



Example: 9

If an object 's downward acceleration is given by $y''(t) = -9.8\, m/s^2$, find the position function $y(t)$. Assume that the initial velocity is $y'(0) = -30\, m/s$ and the initial position is $y(0) = 30{,}000\, m$.

Exercise 9

1) Determine the position function if the velocity function is $v(t) = 3 - 12t\ m/s$ and the initial position is $s(0) = 3\ m$.

2) Determine the position function if the acceleration function is $a(t) = 3\sin t + 1$, the initial velocity is $v(0) = 0$ and the initial position is $s(0) = 4$.

Estimating Distance and Acceleration

Example: 10

The table below shows the velocity of a falling object at different times. For each time interval, estimate the distance fallen and the acceleration.

$t(s)$	0	1.0	2.0	3.0	4.0
$v(t)(m/s)$	0.0	−9.8	−18.6	−24.9	−28.5

Questions from previous MOE exams أسئلة وردت في امتحانات الوزارة السابقة

1

Moe Exam2023/2024

: Understand the notion of indefinite integral as finding an antiderivative

Determine the position function if the acceleration function is $a(t) = t^2 + 1$, the initial velocity is $v(0) = 4$ and the initial position is $s(0) = 0$.

حدّد الدالة المكانية إذا كانت دالة التسارع هي $a(t) = t^2 + 1$، والسرعة المتجهة الابتدائية $v(0) = 4$ والموقع الابتدائي هو $s(0) = 0$.

A	$s(t) = \frac{t^4}{12} + \frac{t^2}{2} + 4t$	B	$s(t) = \frac{t^4}{12} + t^2 + 4t$	C	$s(t) = \frac{t^3}{3} + t + 4$	D	$s(t) = \frac{t^3}{3} + 2t + 4$

A

2

Paper Based كتابى

Moe Exam2022/2023

Determine the position function if the acceleration function is $a(t) = t^2 + 1$, the initial velocity is $v(0) = 4$ and the initial position is $s(0) = 0$.

حدد الدالة المكانية اذا كانت دالة التسارع هي $a(t) = t^2 + 1$, والسرعة المتجهة الابتدائية هي $v(0) = 4$ والموقع الابتدائي هو $s(0) = 0$.

3

Find the antiderivative of a given function

Moe Exam2022/2023

Find the general antiderivative.

$$\int \frac{e^x + 4}{e^x} dx$$

أوجد الدالة الأصلية.

$$\int \frac{e^x + 4}{e^x} dx$$

A	$\ln\|e^x + 4\| + c$	B	$\ln\|e^x\| + c$	C	$x - 4e^{-x} + c$	D	$x - 4e^x + c$

c

4

(B1) إيجاد عكس المشتقة لدالة معطاة 1

Moe Exam2022/2023

Find the general antiderivative.

$$\int 2\sec x \tan x\, dx$$

أوجد الدالة الأصلية.

$$\int 2\sec x \tan x\, dx$$

A	$2\sec x + c$	B	$2\sec^2 x + c$	C	$2\tan^2 x + c$	D	$2\tan x + c$

A

5

Moe Exam2022/2023

(B1) مشتقة كثيرة حدود 1

Find the derivative of the function.

$$f(x) = x^4 + 6x^2 - 2$$

أوجد مشتقة الدالة.

$$f(x) = x^4 + 6x^2 - 2$$

A	$f'(x) = x^5 + 6x^3 - 2x$	B	$f'(x) = x^3 + 6x - 2$	C	$f'(x) = 4x^3 + 12x$	D	$f'(x) = 4x^5 + x^3 - 2x$

C

6

Moe Exam2022/2023

Find the general antiderivative.

$$\int (3x^4 - 3x)dx$$

أوجد الدالة الأصلية.

$$\int (3x^4 - 3x)dx$$

A	$12x^3 - 3 + c$	B	$3x^5 - 3x^2 + c$	C	$x^5 - x^2 + c$	D	$\frac{3}{5}x^5 - \frac{3}{2}x^2 + c$

D

7

Q.10: 111 إيجاد عكس المشتقة لدالة معطاة (B1)

Moe Exam2022/2023

Find the general antiderivative.

أوجد الدالة الأصلية.

$$\int \frac{2x}{x^2+4}dx$$

A	$2x^2(x^2+4)$	B	$\ln\left\|x^2+2x\right\|+c$	C	$(x^2+4)^2+c$	D	$\ln\left\|x^2+4\right\|+c$

D

8

Moe Exam2021/2022

Find the general antiderivative.

أوجد الدالة الأصلية.

$$\int 5\frac{\sin x}{\cos^2 x}dx$$

a. $5\sec^2 x + c$

b. $5\tan^2 x + c$

c. $5\sec x + c$

d. $-5\sec x + c$

C

9

Moe Exam2021/2022

Find the general antiderivative.

أوجد الدالة الأصلية.

$$\int \frac{8x}{x^2+7}dx$$

a. $\frac{1}{2}\ln|x^2+7|+c$

b. $\frac{1}{4}\ln|x^2+7|+c$

c. $2\ln|x^2+7|+c$

d. $4\ln|x^2+7|+c$

D

10

Determine the position function if the velocity function is $v(t) = 8 - 6t$ and the initial position is $s(0) = 4$.

حدّد الدالة المكانية إذا كانت دالة السرعة المتجهة هي $v(t) = 8 - 6t$ والموقع الابتدائي هو $s(0) = 4$.

a. $s(t) = 6t^2 - 8t + 4$

b. $s(t) = 8t - 6t^2 + 4$

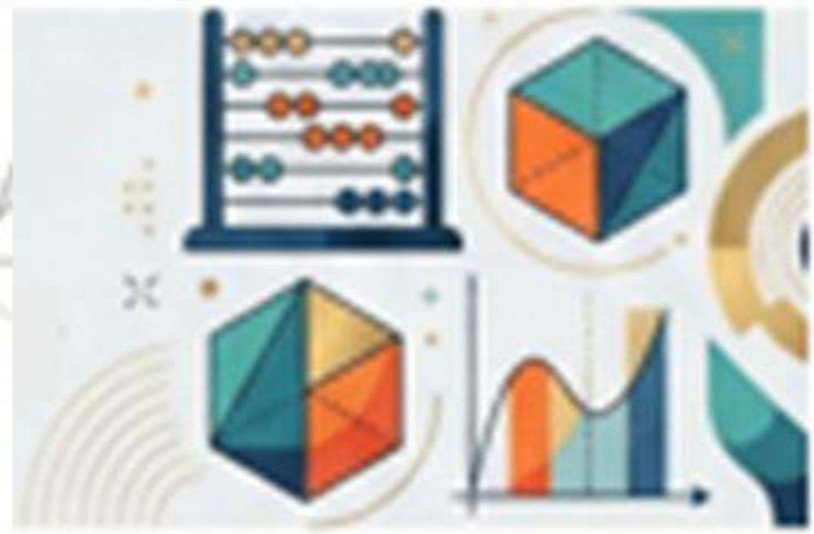
c. $s(t) = 8t - 3t^2 + 4$

d. $s(t) = 3t^2 - 8t + 4$

C

Unit 5: Integration

وحدة 5: التكامل



Lesson 2 : Sums and Sigma notation

الدرس الثاني : المجموع ورمز سيجما

Grade 12 Advanced

الثاني عشر متقدم

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Student Name (اسم الطالب:) ______________________________

Class (الصف) ______________________________

Unit 5	Integration
Lesson 2	Sums and Sigma Notation
Objectives	• Use the sigma notation to compute basic summation.

Sigma Notation

T&S

Sigma Notation is a concise and convenient way to represent long sums.

For example,

$1+2+3+4+5$ (Sum of the first five whole numbers)

$1+4+9+16+25+36$ (Sum of the squares of first six whole numbers)

There is an obvious pattern for numbers involved

If we take a sequence of numbers $a_1, a_2, a_3, \dots, a_n$

then we can write the sum of these numbers(series) as $a_1 + a_2 + a_3 + \cdots + a_n$

A shorter way of writing this:

Sigma Notation

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \cdots + a_n$$

- Upper limit (Last value of i) → n
- Greek letter "Sigma" Summation → $\sum$
- General term of the sequence → a_i
- Index of Summation → i
- Lower limit (First value of i) → 1

Example: 1

Write in summation notation:

$a)\ \sqrt{1}+\sqrt{2}+\sqrt{3}+\cdots+\sqrt{10}$

$b)\ 3^3+4^3+5^3+\cdots+45^3$

Solution

Exercise: 1.1

Translate into summation notation $Q1)\ \ 2(1)^2 + \ 2(2)^2 + \ 2(3)^2 + \cdots + \ 2(14)^2$

$Q2)\ \sqrt{2-1} + \sqrt{3-1} + \sqrt{4-1} + \cdots + \sqrt{15-1}$

Solution

Example: 2

Write in summation notation: The sum of the first 200 odd positive integers.

Solution

Example: 3

Write out all terms and compute the sums.

$a)\ \sum_{i=1}^{8}(2i+1)$

$b)\ \sum_{i=2}^{6} \boldsymbol{sin}(\boldsymbol{2\pi i})$

$c)\ \sum_{i=1}^{8} 5$

Exercise: 3.1

Write out all terms and compute the sums.

$a)\ \sum_{i=1}^{6} 3i^2$

$b)\ \sum_{i=6}^{10} (4i+2)$

THEOREM 2.1

If n is any positive integer and c is any constant, then.

$(i)\ \sum_{i=1}^{n} c = cn$ Sum of constants

$(ii)\ \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ Sum of the first n positive integers

$(iii)\ \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ Sum of the squares of the first n positive integers

THEOREM 2.2

For any constants c and d,

$$\sum_{i=1}^{n} (ca_i + db_i) = c\sum_{i=1}^{n} a_i + d\sum_{i=1}^{n} b_i$$

Example: 4

Compute: (Use summation rules to compute the sum):

$a)\ \sum_{i=1}^{8}(2i+1)$

$b)\ \sum_{i=1}^{800}(2i+1)$

Exercise: 4.1

Use summation rules to compute the sum:

$\boldsymbol{a})\ \sum_{\boldsymbol{i=1}}^{\boldsymbol{70}}(\boldsymbol{3i-1})$

$\boldsymbol{b})\ \sum_{\boldsymbol{i=1}}^{\boldsymbol{50}}(\boldsymbol{8-i})$

Example: 5

Compute:

$a)\ \sum_{i=1}^{20} i^2$

$b)\ \sum_{i=1}^{20}\left(\frac{i}{20}\right)^2$

Exercise: 5.1

Use summation rules to compute the sum:

a) $\sum_{n=1}^{100}(n^2 - 3n + 2)$

b) $\sum_{i=7}^{20} 5$

c) $\sum_{k=0}^{n}(k^2 + 5)$

Example: 6 Sum the values of $f(x) = x^2 + 3$, evaluated at $x = 0.1,\ x = 0.2,\ x = 0.3, \ldots, x = 1.0$

Example: 7 Compute the sum of the form $\sum_{i=1}^{n} f(x_i)\Delta x$ for the given values of x_i

$f(x) = x^2 + 4x,\ \ x = 0.2, 0.4, 0.6, 0.8, 1.0 \qquad \Delta x = 0.2; n = 5$

Questions from previous MOE أسئلة وردت في امتحانات الوزارة السابقة

Question 3 MOE Exam 2023/2024

Q.3: Use the sigma notation to compute basic summation

Compute the sum.

احسب المجموع.

$$\sum_{i=6}^{10}(i+4)$$

A	60	B	95	C	40	D	220

Question 14 MOE Exam 2021/2022

Compute the sum.

احسب المجموع.

$$\sum_{i=5}^{9}(i^2+3)$$

A	42
B	70
C	312
D	270

Final Revision Grade 12 Adv

الأستاذ عبدالله أبو النجا معلم أول الرياضيات 0505114830

Question 18 MOE Exam 2022/2023

18

Q.18: 1 استخدام رمز المجموع سيجما لإيجاد المجاميع البسيطة (B2)

Compute the sum of the form

$$\sum_{i=1}^{n} f(x_i)\Delta x$$

for values x_i where

$f(x) = 3x + 5;\ x = 2, 4, 6$

$n = 3;\ \Delta x = 2.$

احسب المجموع بالصيغة.

$$\sum_{i=1}^{n} f(x_i)\Delta x$$

لقيم x_i حيث

$f(x) = 3x + 5;\ x = 2, 4, 6$

$.n = 3;\ \Delta x = 2$

A 22

B 51

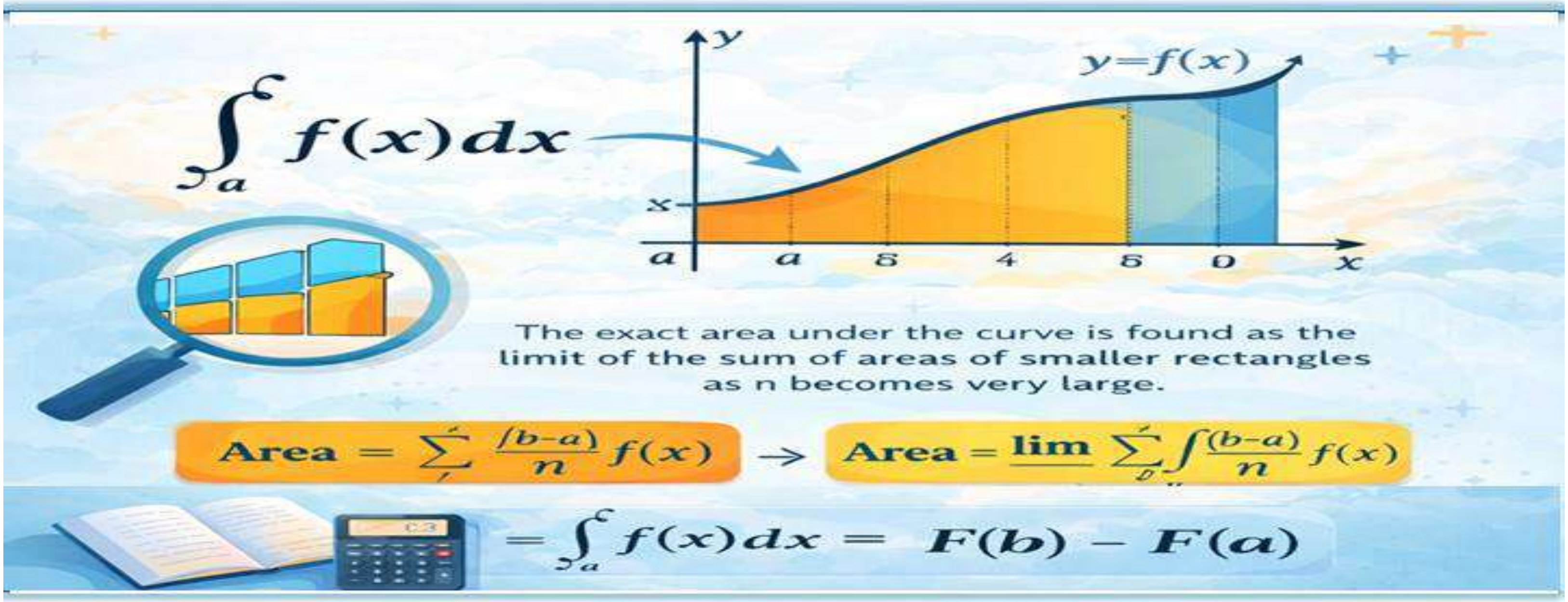
C 102

D 11

Mr. Abdalla Abouelnaga | عزيزي الطالب : اجتهد في فهم ومذاكرة هذه الأمثلة والتمارين بالكتاب المدرسي تمنياتي للجميع بالنجاح والتفوق | Solution Steps Before Choosing

Unit 5: Integration

وحدة 5: التكامل



Lesson 3 : Area

الدرس الثالث :المساحة تحت منحنى

Grade 12 Advanced

الثاني عشر متقدم

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Class (الصف) ____________________

Unit 5	Integration
Lesson 3	Area

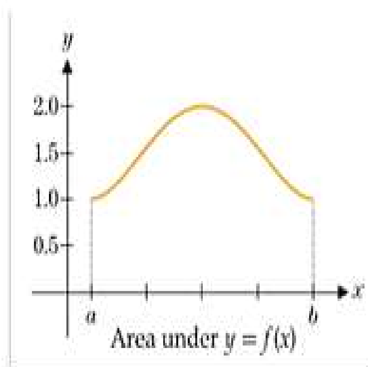
Objectives

- Estimate the area under a curve on a given interval using rectangles.
- Compute the area under a curve using summations and limits.

Area

To compute the area beneath the graph of $y = f(x)$ and above $x - axis$ on the interval $[a, b]$

For $f(x) \geq 0$ and f is continuous on interval $[a, b]$



To approximate the area:

Partition: Divide the interval $[a, b]$ into n subintervals

Length of each subinterval: $\Delta x = \dfrac{b-a}{n}$

The points are: $x_0 = a$

$x_1 = x_0 + \Delta x$

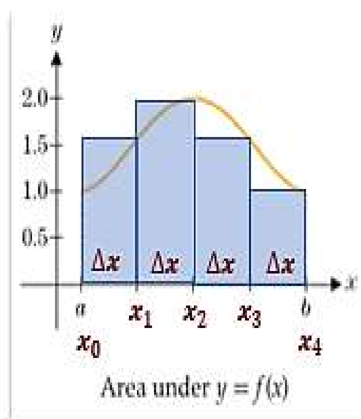
$x_2 = x_1 + \Delta x$ And so on $\longrightarrow$ $x_i = a + i\Delta x$, $i = 1, 2, 3, \ldots, n$

Construct a rectangle on each subinterval $[x_{i-1}, x_i]$

Width: Δx

Length: $f(x_i)$ The value of the function at the **right endpoint** of the subinterval

Then the area beneath the graph of $y = f(x)$ is roughly the sum of the areas of rectangles.



Area: $A \approx$ Sum of the areas of rectangles

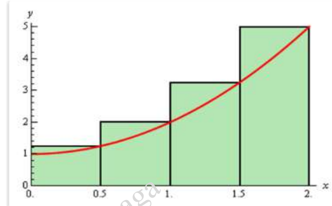
Area : $A \approx f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + \cdots + f(x_n)\Delta x$

$$A \approx A_n = \sum_{i=1}^{n} f(x_i)\Delta x$$

Let $n = 4$

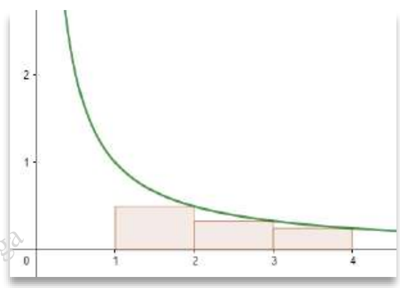
Area $= A \approx A_4 = f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x$

Example: 1 Approximate the area under the curve $y = f(x) = x^2 + 1$ on the interval $[0,2]$, Using 4 rectangles.



Solution

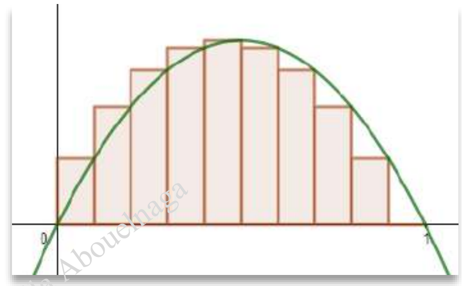
Exercise: 1.1 Approximate the area under the curve $y = f(x) = \frac{1}{x}$ on the interval $[1,4]$, Using 3 rectangles.



Solution

Exercise: 1.2 Approximate the area under the curve $y = f(x) = 2x - 2x^2$ on the interval $[0,1]$, Using 10 rectangles.

Solution



DEFINITION 3.1

For a function f defined on interval $[a, b]$, if f is continuous on $[a, b]$ and $f(x) \geq 0$ on $[a, b]$, the area A under the curve $y = f(x)$on $[a, b]$ is given by

$$A = \lim_{n\to\infty} A_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

Example: 2 Find the area under the curve $y = f(x) = 2x - 2x^2$ on the interval $[0,1]$.

Solution

Exercise: 2.1 Estimate the area under the curve $y = f(x) = \sqrt{x + 1}$ on the interval $[1,3]$.

Solution

DEFINITION 3.2

Let $\{x_0, x_1, x_2, \dots\dots, x_n\}$ be a regular partition of the interval $[a, b]$,
with $x_i - x_{i-1} = \Delta x = \frac{b-a}{n}$, For all i.
Pick points $c_1, c_2, \dots\dots, c_n$, where c_i is any point in the subinterval $[x_{i-1}, x_i]$, For $i = 1,2, \dots, n$.
(These called evaluation points).
The Riemann Sum for this partition and set of evaluation points is: $\displaystyle\sum_{i=1}^{n} f(c_i)\Delta x$

For a continuous, nonnegative function f, the area under the curve $y = f(x)$ is the limit of Riemann sum.

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x \qquad \text{Where } c_i = x_i \text{ (right end point), For } i = 1,2, \dots, n$$

The limit will give the same answer for any evaluation point $c_i \in [x_{i-1}, x_i]$, For $i = 1,2, \dots, n$

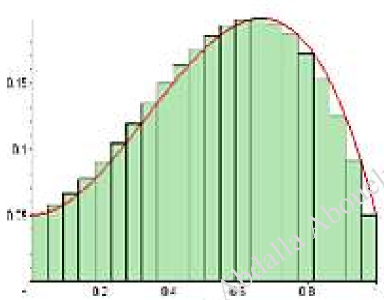
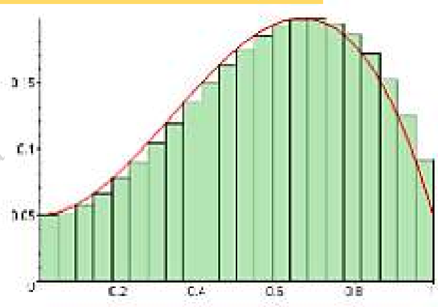
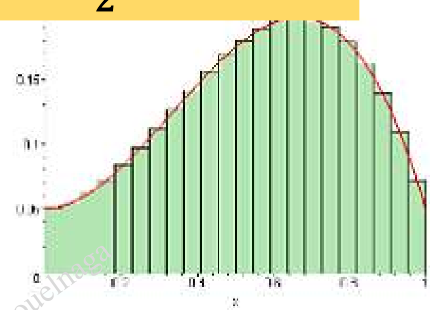
Choice of Evaluation Point

When we cannot compute the limit of Riemann sums (or at least not directly).

We can approximate the area by calculating Riemann sums for large values of n

Choices for evaluation points $\quad c_i \in [x_{i-1}, x_i]$, For $i = 1, 2, \dots, n$

The Right Endpoint	The Left Endpoint	The Mid Endpoint
$c_i = x_i$	$c_i = x_{i-1}$	$c_i = \frac{1}{2}(x_{i-1} + x_i)$

Choice of Evaluation Point

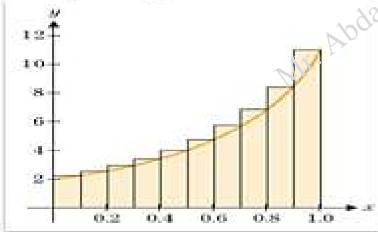
$c_i \in [x_{i-1}, x_i]$, For $i = 1, 2, \dots, n$

For Increasing function

Example: $f(x) = 9x^2 + 2$, on $[0,1]$

The Right Endpoint

$c_i = x_i$

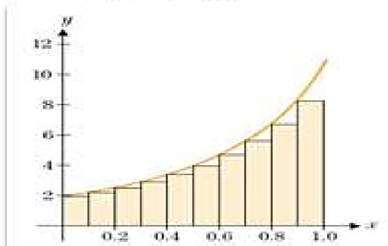


Gives overestimation for the area

$\text{approximated area} > \text{Actual area under graph}$

The Left Endpoint

$c_i = x_{i-1}$

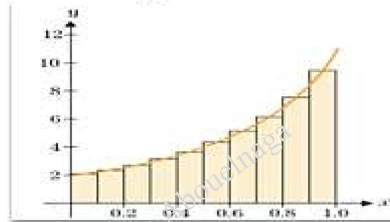


Gives underestimation for the area

$\text{approximated area} < \text{Actual area under graph}$

The Midpoint

$c_i = \frac{1}{2}(x_{i-1} + x_i)$



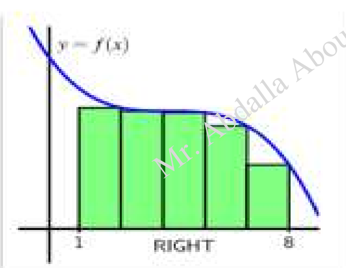
Choice of Evaluation Point

$c_i \in [x_{i-1}, x_i]$, For $i = 1, 2, \dots, n$

For decreasing function

The Right Endpoint

$c_i = x_i$

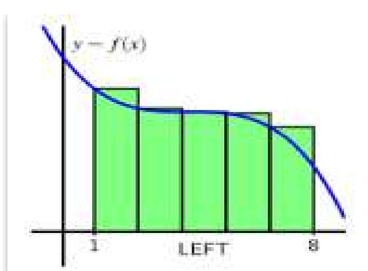


Gives underestimation for the area

$\text{approximated area} < \text{Actual area under graph}$

The Left Endpoint

$c_i = x_{i-1}$

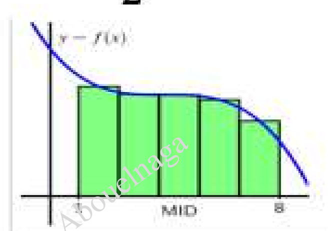


Gives overestimation for the area

$\text{approximated area} > \text{Actual area under graph}$

The Midpoint

$c_i = \frac{1}{2}(x_{i-1} + x_i)$



$y = f(x)$ on $[a, b]$ $\Delta x = \frac{b-a}{n}$ $x_i = a + i\Delta x$

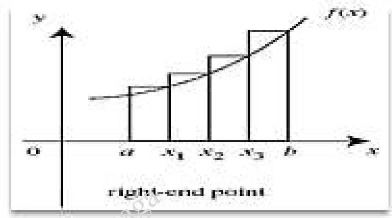
$c_i \in [x_{i-1}, x_i]$, For $i = 1, 2, \dots, n$

The Right Endpoint $c_i = x_i$

Evaluation points: $c_i = a + i\Delta x$

$i = 1, 2, 3, \dots, n$

Riemann Sum: $A_n = \sum_{i=1}^{n} f(c_i)\Delta x$

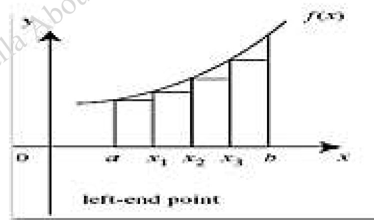


The Left Endpoint $c_i = x_{i-1}$

Evaluation points: $c_i = a + (i-1)\Delta x$

$i = 1, 2, 3, \dots, n$

Riemann Sum: $A_n = \sum_{i=1}^{n} f(c_i)\Delta x$

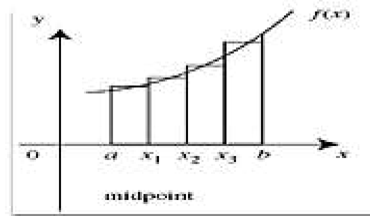


The Midpoint $c_i = \frac{1}{2}(x_{i-1} + x_i)$

Evaluation points $c_i = a + \left(i - \frac{1}{2}\right)\Delta x$

$i = 1, 2, 3, \dots, n$

Riemann Sum: $A_n = \sum_{i=1}^{n} f(c_i)\Delta x$



Example: 3

Approximate the area under the curve $y = f(x) = x^2 + 1$ on the interval $[0,1]$, using $n = 16$ rectangles.

And the evaluation rules a) left endpoint b) midpoint c) right endpoint.

Solution

a) left endpoint

b) midpoint

c) right endpoint.

Example: 4

Compute Riemann sum for $y = f(x) = \sqrt{x+1}$ on the interval $[1,3]$, For $n = 10, 50, 100, 500,\ 1000 and\ 5000$. Using

a) left endpoint b) midpoint c) right endpoint. Of each interval as the evaluation point

Solution

a) left endpoint

b) midpoint

c) right endpoint.

Example 5

Use Riemann sum and a limit to compute the exact area under the curve $y = f(x) = x^2 + 3x$ on the interval $[0,1]$.

Solution

Example 6 Use the given function values in the table below to estimate the area under the curve using left-endpoint and right-endpoint evaluation.

Solution

Left-Endpoint

	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
x	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
$f(x)$	2.0	2.4	2.6	2.7	2.6	2.4	2.0	1.4	0.6

Right-Endpoint

	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
x	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
$f(x)$	2.0	2.4	2.6	2.7	2.6	2.4	2.0	1.4	0.6

Question 15 MOE Exam 2023/2024

Q.15: Estimate the area under a curve on a given interval using rectangles

Use the given function values to estimate the area under the curve using right-endpoint evaluation.

استخدم قيم الدالة المعطاة لتقدير المساحة تحت المنحنى باستخدام قيم نقطة النهاية اليمنى.

x	0.0	0.2	0.4	0.6	0.8
$f(x)$	2.0	2.2	1.6	1.4	1.6

A	B	C	D
1.36	1.44	6.8	7.2

Question 17 MOE Exam 2022/2023

17

Q.17: 1 تقدير المساحة تحت المنحنى لدالة في فترة محددة باستخدام المستطيلات (B2)

Use the given function values to estimate the area under the curve using right-endpoint evaluation.

استخدم قيم الدالة المعطاة لتقدير المساحة تحت المنحنى باستخدام قيم نقطة النهاية اليمنى.

x	1.0	1.2	1.4	1.6	1.8
$f(x)$	0.0	0.4	0.6	0.8	1.2

A 0.6

B 0.36

C 3

D 4.76

Question 15 MOE Exam 2021/2022

Use the given function values to estimate the area under the curve using left-endpoint evaluation.

استخدم قيم الدالة المعطاة لتقدير المساحة تحت المنحنى باستخدام قيم نقطة النهاية اليسرى.

x	0.0	0.1	0.2	0.3	0.4
$f(x)$	2.0	2.4	2.6	2.7	2.6

A 1.03

B 9.7

C 0.97

D 10.3

Question 4 MOE Exam 2023/2024

Question 4	Compute the area under a curve using summations and limits إيجاد المساحة تحت المنحنى لدالة باستخدام المجاميع والنهايات	السؤال 4

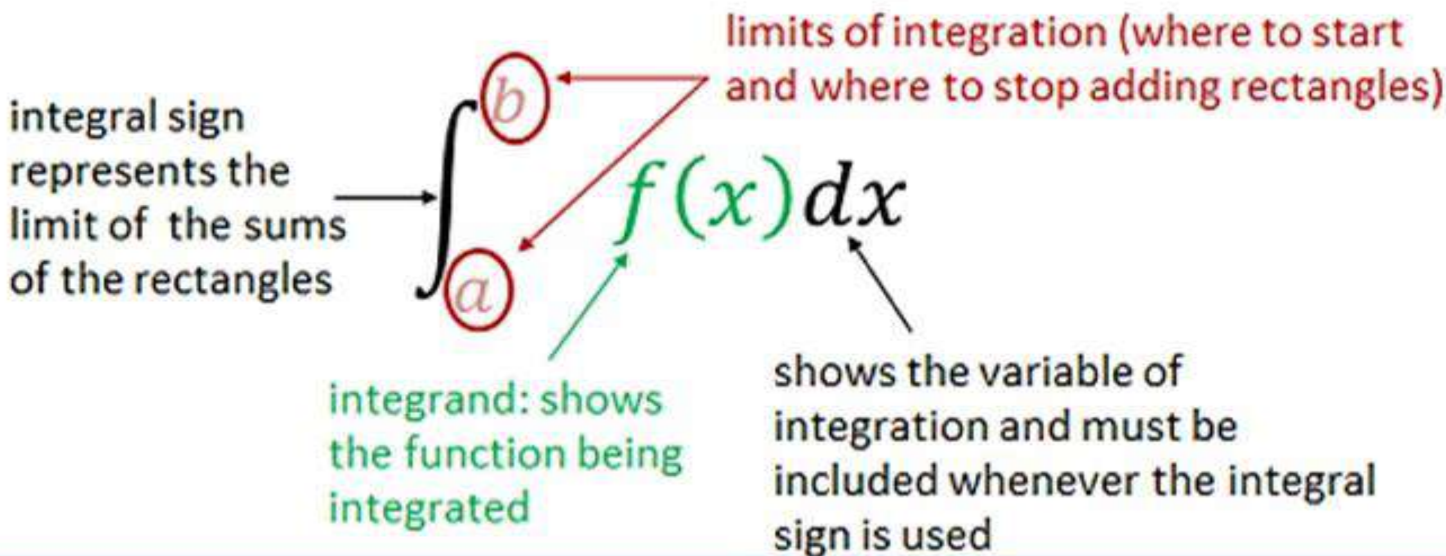
Use $A = \lim_{n\to\infty} A_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$, to find the exact area under the curve $y = f(x) = x^2 + 1$ on the interval $[0, 1]$.

استخدم $A = \lim_{n\to\infty} A_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$، لإيجاد المساحة الدقيقة تحت المنحنى $y = f(x) = x^2 + 1$ على الفترة $[0, 1]$.

Solution

Unit 5: Integration

وحدة 5: التكامل



Lesson 4 : The Definite integral

الدرس الرابع :التكامل المحدود

Grade 12 Advanced

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Class (الصف) ____________________

Unit 5	Integration
Lesson 4	The Definite Integral

Objectives

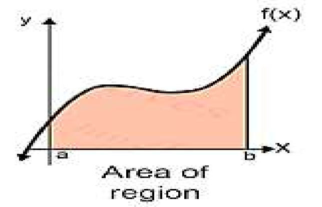
- Understand the notion of a definite integral.
- Compute a definite integral using Riemann sums.
- Find the area under a curve on a given interval using Riemann sums.
- Learn the properties of definite integrals.
- Apply the Integral Mean Value Theorem.

Area Under a Curve

The limit of Riemann Sums

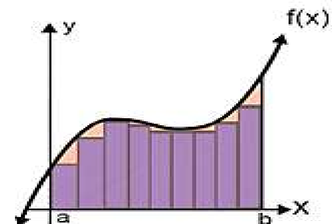
$f(x) \geq 0$ on $[a, b]$

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x$$



For any $c_i \in [x_{i-1}, x_i]$, *For* $i = 1, 2, \ldots, n$

What if the values of $f(c_i)$ *are negative* ?



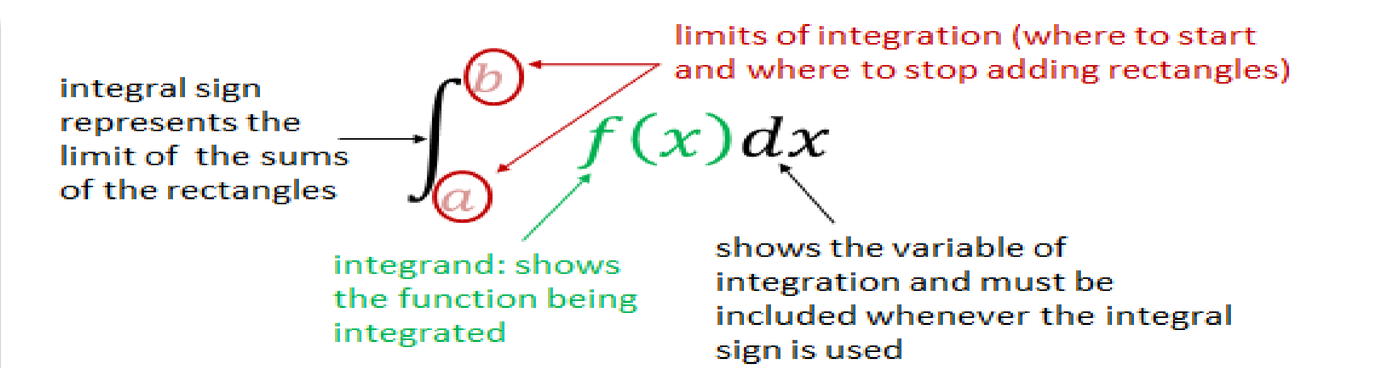
Definition 4.1

For any function f defined on $[a, b]$, the **definite integral** of f from a to b is

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x,$$

Whenever the limit exists and is the same for every choice of evaluation points, $c_1, c_2, \ldots.. c_n$.
When the limit exists, we say that f is **integrable** on $[a, b]$.

Definite Integral



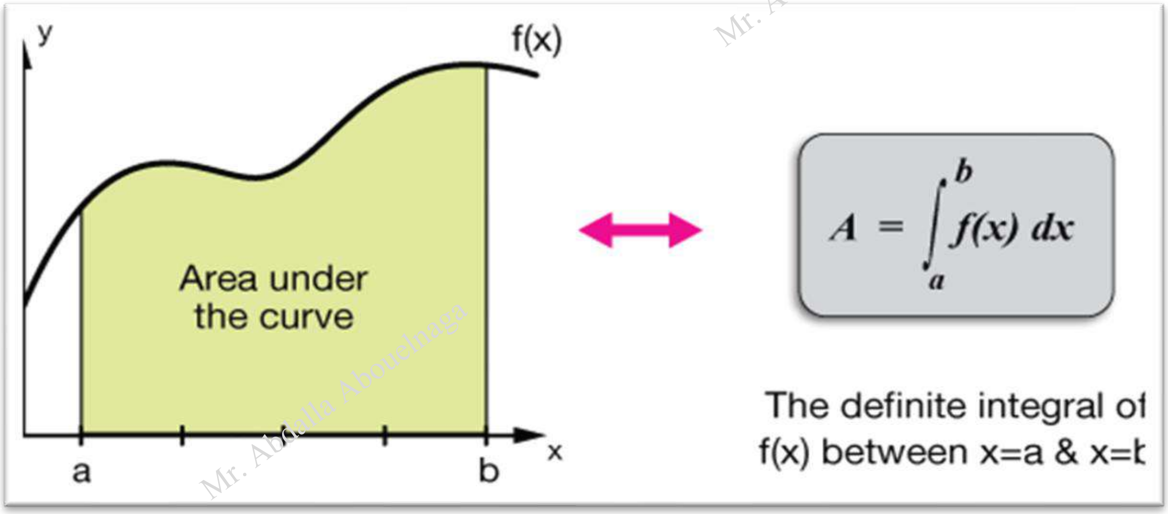
Theorem 4.1

If f is **continuous** on the closed interval $[a,b]$, then f is **integrable** on $[a,b]$.

Note:

If f is continuous on the closed interval $[\boldsymbol{a},\boldsymbol{b}]$ and $\boldsymbol{f(x)\geq 0}$ on $[\boldsymbol{a},\boldsymbol{b}]$ then

$\int_a^b f(x)dx$ =*Area under the curve*



REMEMBER

Choice of Evaluation Point

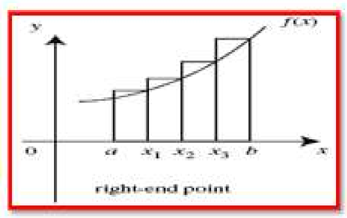
$y=f(x)$ on $[a,b]$ $\qquad \Delta x=\dfrac{b-a}{n}$ $\qquad x_i=a+i\Delta x$

$c_i \in [x_{i-1},x_i]$, For $i=1,2,\dots,n$

The Right Endpoint $c_i=x_i$

Evaluation points: $c_i=a+i\Delta x$, $i=1,2,3,\dots,n$

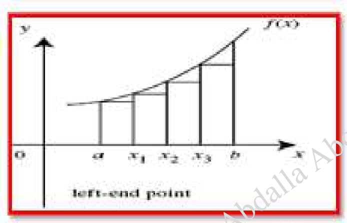
Riemann Sum: $A_n=\sum_{i=1}^{n} f(c_i)\Delta x$



The Left Endpoint $c_i=x_{i-1}$

Evaluation points: $c_i=a+(i-1)\Delta x$, $i=1,2,3,\dots,n$

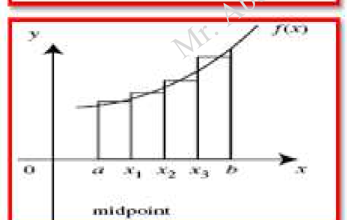
Riemann Sum: $A_n=\sum_{i=1}^{n} f(c_i)\Delta x$



The Midpoint $c_i=\dfrac{1}{2}(x_{i-1}+x_i)$

Evaluation points: $c_i=a+\left(i-\dfrac{1}{2}\right)\Delta x$, $i=1,2,3,\dots,n$

Riemann Sum: $A_n=\sum_{i=1}^{n} f(c_i)\Delta x$



Example: 1

Use the midpoint rule with $n = 6$ to estimate: $\int_0^3 \sqrt{x^2+1}\,dx$

Solution

Exercise: 1.1

Use the midpoint rule to estimate: $\int_0^{15} 30\left(1-e^{-x/3}\right)dx$

Solution

Example: 2

Evaluate the integral by computing the limit of Riemann sums: $\int_0^1 2x\,dx$

Solution

Exercise: 2.1

Evaluate the integral by computing the limit of Riemann sums: $\int_1^3 (x^2-3)\,dx$

Solution

A Riemann Sum for a Function with Positive and Negative Values

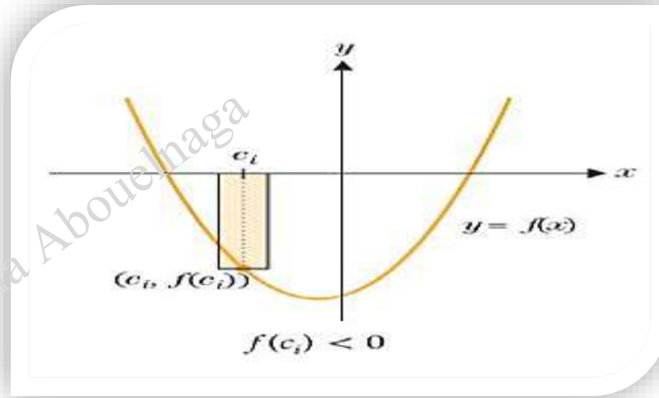
When $\boldsymbol{f}$ is both negative and positive on $[\boldsymbol{a}, \boldsymbol{b}]$

When calculating Riemann sum

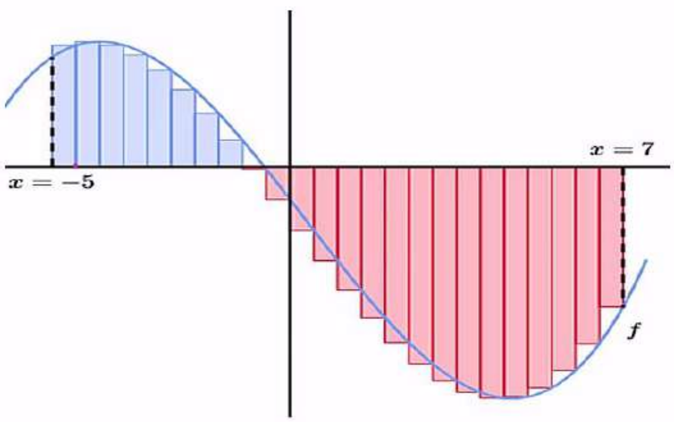
If $\boldsymbol{f(c_i) < 0}$ $\quad$ $\boldsymbol{for\ some\ i}$

The height of rectangle is negative.

$\boldsymbol{f(c_i)\Delta x = -area\ of\ the\ ith\ rectangle}$



The graph of f is shown below. A total of 24 right-hand rectangles are shown, eight in blue and 16 in red. All 24 of the rectangles have the same width. Which of the statements below is/are **true**?



I. $\sum_{i=1}^{8} f\left(-5 + \frac{i}{2}\right) \cdot \frac{1}{2}$ is the sum of the areas of the blue rectangles.

II. $\sum_{i=1}^{16} f\left(-1 + \frac{i}{2}\right) \cdot \frac{1}{2}$ is the sum of the areas of the red rectangles.

III. $\sum_{i=1}^{24} f\left(-5 + \frac{i}{2}\right) \cdot \frac{1}{2}$ is the sum of the areas of all of the rectangles.

Definition 4.2

Suppose that $f(x) \geq 0$ on the interval $[a, b]$ and A_1 is the area bounded between the curve $y = f(x)$ and the $x - axis$ for $a \leq x \leq b$.

Further, suppose that $f(x)$ on the interval $[b, c]$ and A_2 is the area bounded between the curve $y = f(x)$ and the $x - axis$ for $b \leq x \leq c$.

The signed area between $y = f(x)$ and the $x - axis$ for $a \leq x \leq c$ is:

$$Signed\ Area = A_1 - A_2$$

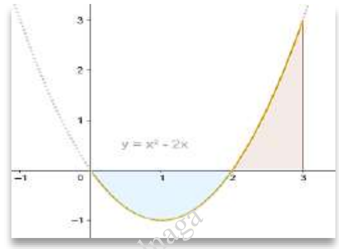
$$\int 1dx + \int 2dx$$

The Total area between $y = f(x)$ and the $x - axis$ for $a \leq x \leq c$ is:

$$Total\ Area = A_1 + A_2$$

$$\int 1dx - \int 2dx$$

Example: 3 Compute the integrals and interpret each in terms of area:



$a)\ \int_0^2 (x^2 - 2x)dx$ $b)\ \int_0^3 (x^2 - 2x)dx$

Solution

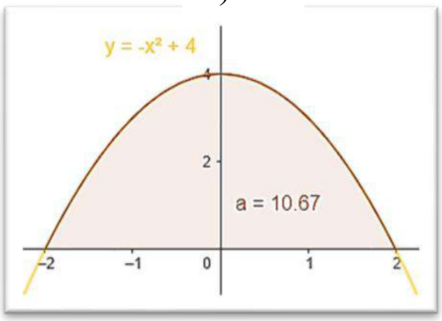
$a)\ \int_0^2 (x^2 - 2x)dx$

$b)\ \int_0^3 (x^2 - 2x)dx$

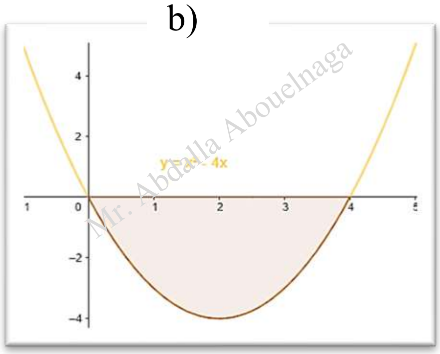
Example: 4

Write the given total area as an integral

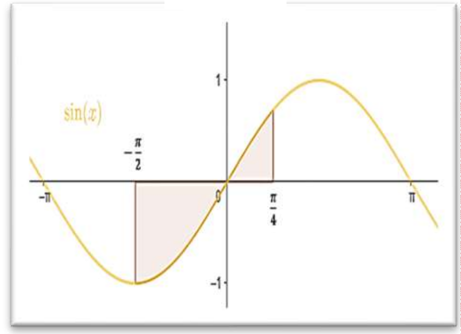
a)



b)



c)

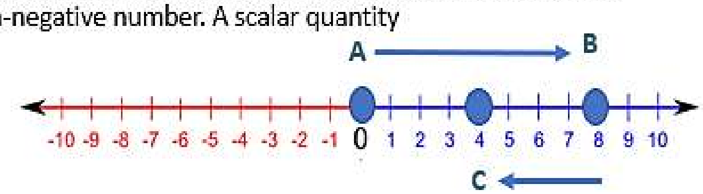


Solution

Estimating Overall Change in Position

Distance Total amount the object has moved. This depends on the whole path traveled, not just the starting and ending points. Distance traveled is always a non-negative number. A scalar quantity

Displacement Change in position of an object.



An object moved from A to B then moved to C

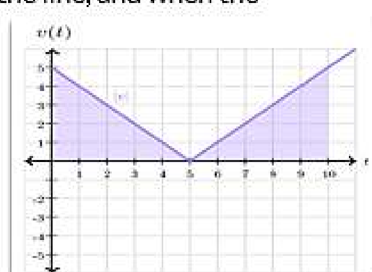
Find **Total Distance** $= 8 + 4 = 12\ units$

Change in posit $= final\ position\ - initial\ position$

$= 4 - 0 = 4\ units$ As the object move 8 units forward then it moved 4 units backward $+8 - 4 = 4$

$final\ position($object's position$)$ = **Change in position** + $initial\ position$

Velocity When the velocity is positive it means the particle is moving forward along the line, and when the velocity is negative it means the particle is moving backwards.



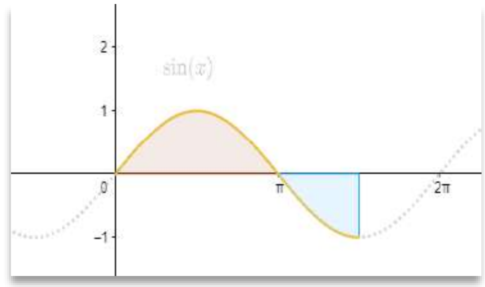
Change in position on $[0, T]$ $= \int_0^T v(t)dt$ **Signed Area**

Total Distance $= \int_0^T |v(t)|dt$ **Total Area**

We are going to evaluate the area of each region then we will add areas

Example: 5

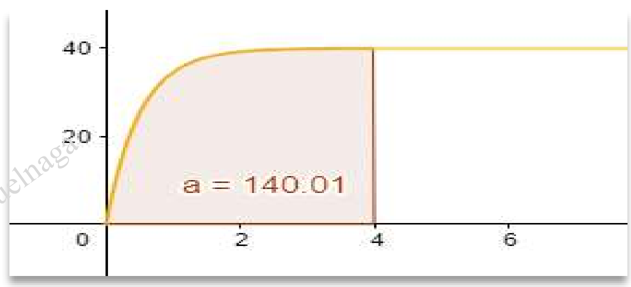
An object moving along a straight line has velocity function $v(t) = sint$, if the object starts at position 0, determine the total distance travelled and the object's position at time $t = 3\pi/2$



Solution

Exercise: 5.1

Use the given velocity function and initial position to estimate the final position $s(b)$. $v(t) = 40(1 - e^{-2t})$, $s(0) = 0$, $b = 4$



Solution

Theorem 4.2

If f and g are integrable on $[a,b]$, then the following are true.

i. For any constant c and d,

$\int_a^b [cf(x) + dg(x)]dx = c\int_a^b f(x)dx + d\int_a^b g(x)dx$

ii. For any c in $[a,b]$,

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^d f(x)dx$$

Definitions:

For any integrable function f, If $a < b$ we define:

$$\int_b^a f(x)dx = -\int_a^b f(x)dx$$

*If we integrate **backward** along an interval, the width of the rectangles corresponding to a Riemann sum Δx would seem to be **negative**.*

If $f(a)$ is defined, we define:

$$\int_a^a f(x)dx = 0$$

*Definite integral represents **area**, area from a to a is **zero**.*

Example: 6 Use theorem 4.2 to write the expression as a single integral.

a) $\int_0^2 f(x)dx + \int_2^3 f(x)dx =$

b) $\int_0^3 f(x)dx - \int_2^3 f(x)dx =$

c) $\int_0^2 f(x)dx + \int_2^1 f(x)dx =$

d) $\int_{-1}^2 f(x)dx + \int_2^3 f(x)dx =$

Example: 7

Assume that $\int_1^3 f(x)dx = 3$ and $\int_1^3 g(x)dx = -2$

Find

$a) \displaystyle\int_1^3 [f(x) + g(x)]dx =$

$b) \displaystyle\int_1^3 [2f(x) - g(x)]dx =$

c) $\displaystyle\int_1^3 [f(x) - g(x)]dx =$

$d) \displaystyle\int_1^3 [4f(x) - 3g(x)]dx =$

Piecewise Continuous Functions:

Continuous functions on a closed interval are integrable on that interval. *Theorem 4.1*

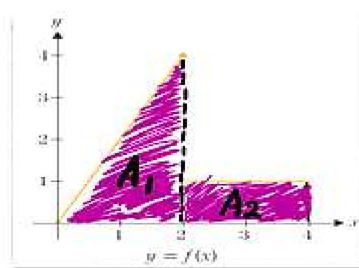
Discontinuous function with finite number of jump discontinuities (piecewise continuous function) are integrable.

Example:

The function $\boldsymbol{f(x)}$ *has a jump discontinuity at* $\boldsymbol{x = 2}$

Otherwise, the function is continuous

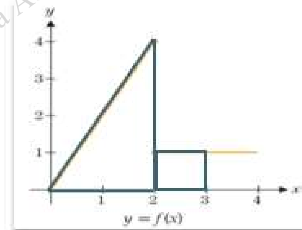
$$\int_0^4 f(x)dx = \int_0^2 f(x)dx + \int_2^4 f(x)dx$$
$$= A_1 + A_2$$



An Integral with a Discontinuous Integrand

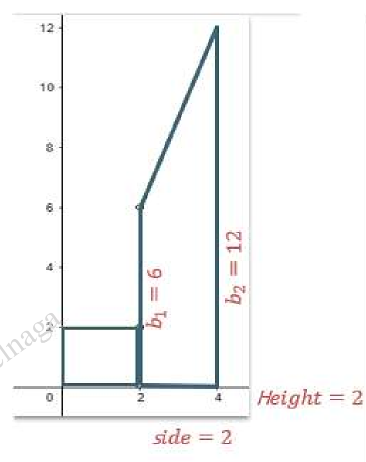
Example: 8

Evaluate: $\int_0^3 f(x)dx$, *where* $f(x)$ *is denoted by* $f(x) = \begin{cases} 2x, & if\ x \le 2 \\ 1, & if\ x > 2 \end{cases}$



Exercise: 8.1

Evaluate: $\int_0^4 f(x)dx$, *where* $f(x)$ *is denoted by* $f(x) = \begin{cases} 2, & if\ x \le 2 \\ 3x, & if\ x > 2 \end{cases}$



Exercise: 8.2

Use geometric formula to compute the integral:

a) $\int_0^2 3x\,dx$

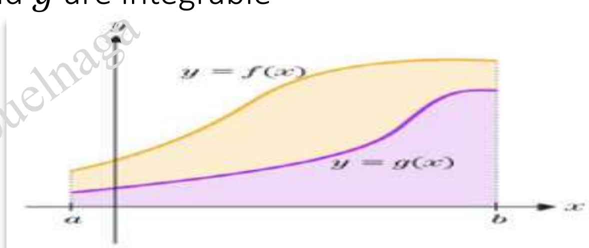
b) $\int_1^4 2x\,dx$

c) $\int_0^2 \sqrt{4-x^2}\,dx$

Theorem 4.3

Suppose that $g(x) \le f(x)$ for all $x\epsilon[a,b]$ and that f and g are integrable on $[a,b]$. Then,

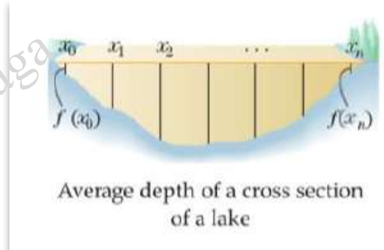
$$\int_a^b g(x)dx \le \int_a^b f(x)dx$$



Larger functions have larger integrals

Average Value of a Function

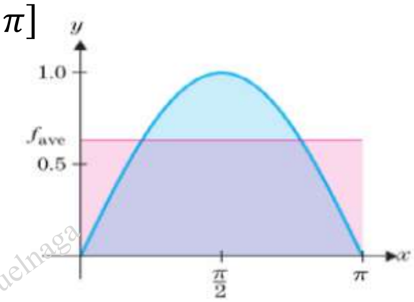
$$f_{ave} = \lim_{n\to\infty}\left[\frac{1}{b-a}\sum_{i=1}^{n} f(x_i)\Delta x\right] = \frac{1}{b-a}\int_a^b f(x)dx$$



Average depth of a cross section of a lake

Example: 9

Compute the average value of: $f(x) = sinx \text{ on the interval } [0,\pi]$



Exercise: 9.1

Compute the average value of: $f(x) = 2x + 1 \text{ on the interval } [0,4]$

Exercise: 9.2 Compute the average value of: $f(x) = x^2 + 2x$ *on the interval* $[0,1]$

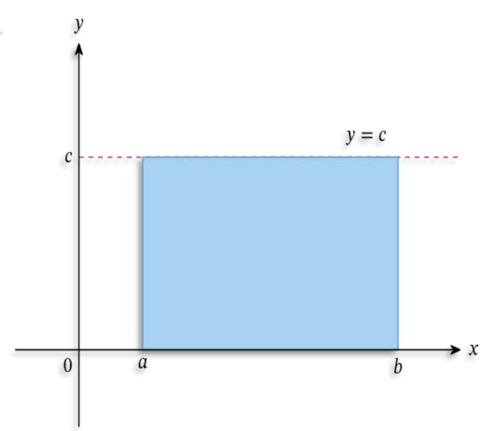
Definite Integral of Constant Function

$$\int_a^b c\,dx \quad = c(b-a)$$

Or $\int_a^b c\,dx$ $= \textit{area of rectangle}$

$= \textit{length} \times \textit{width}$

$= c(b-a)$



Estimating the value of Definite Integral

Let f be a continuous function on a closed interval $[a,b]$

Then, f has a maximum and minimum value

Extreme Value Theorem
Let M : the maximum value
Let m : the minimum value

Then, $m \le f(x) \le M \qquad x\epsilon[a,b]$ *Using Theorem 4.3*

$$\int_a^b m\,dx \le \int_a^b f(x)dx \le \int_a^b M\,dx$$ *m, M are constants*

Integral Mean Value Theorem

$$m(b-a) \le \int_a^b f(x)dx \le M(b-a)$$ *Divide by $b-a$*

$$m \le \frac{1}{b-a}\int_a^b f(x)dx \le M$$ *Average value of f on $[a,b]$*

Average value lies between the maximum and minimum , since f is a continuous function

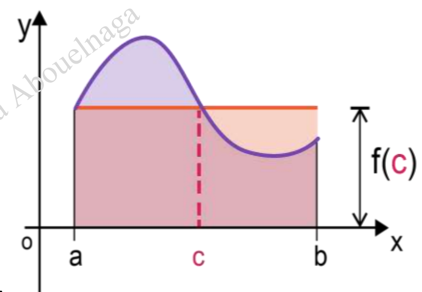
There must be some value $c\epsilon(a,b)$ for which

$$f(c) = \frac{1}{b-a}\int_a^b f(x)dx$$

Theorem 4.4(Integral Mean Value Theorem)

If f is continuous on $[a,b]$, then there is a number $c\epsilon(a,b)$ for which.

$$f(c) = \frac{1}{b-a}\int_a^b f(x)dx$$



Continuous function will take on its average value at some point.

This is only another form of differential Mean Value Theorem

Example: 10

Use integral Mean Value Theorem to Estimate the value of:

$$\int_0^1 \sqrt{x^2+1}\,dx$$

Remember
If $m \le f(x) \le M \qquad x\,\epsilon\,[a,b]$
Then,
$m(b-a) \le \int_a^b f(x)dx \le M(b-a)$

Exercise: 10.1 Use Integral Mean Value Theorem to Estimate the value of:

a) $\int_{\pi/3}^{\pi/2} 3\cos x^2 \, dx$

b) $\int_{-1}^{1} \frac{3}{x^3+2} dx$

Example: 11 Find the value of c that satisfy the conclusion of the Integral Mean Value Theorem

$$\int_0^2 3x^2 \, dx \quad (= 8)$$

Exercise: 11.1

Find the value of c that satisfy the conclusion of the Integral Mean Value Theorem $\int_{-1}^{1}(x^2-2x)\,dx \quad \left(=\frac{2}{3}\right)$

Question 4 MOE Exam 2023/2024

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Q.4: Apply the Integral Mean Value Theorem

Compute the average value of $f(x) = 3x^2 - 1$ on the interval $[0, 2]$.

احسب القيمة المتوسطة لـ $f(x) = 3x^2 - 1$ على الفترة $[0, 2]$.

A		B		C		D	
A	3	B	5	C	6	D	10

Question 7 MOE Exam 2023/2024

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Q.7: Learn the properties of definite integrals

Evaluate $\int_0^3 f(x)dx$, where

$$f(x) = \begin{cases} 4x, & x \le 2 \\ 1, & x > 2 \end{cases}$$

أوجد قيمة $\int_0^3 f(x)dx$، حيث

$$f(x) = \begin{cases} 4x, & x \le 2 \\ 1, & x > 2 \end{cases}$$

A	6	B	9	C	16	D	21

Question 14 MOE Exam 2022/2023

14

Q.14: 1 التعرف على خصائص التكامل المحدود (B2)

Assume that:

$\int_2^4 f(x)dx = -5$ and $\int_2^4 g(x)dx = 3$.

find $\int_2^4 [4g(x) - 3f(x)]dx$.

فرضًا أنّ:

$\int_2^4 g(x)dx = 3$ و $\int_2^4 f(x)dx = -5$.

أوجد $\int_2^4 [4g(x) - 3f(x)]dx$.

A. 27

B. -3

C. 1

D. 2

Question 15 MOE Exam 2022/2023

15

Q.15: 1 تطبيق نظرية القيمة المتوسطة في التكامل (B2)

Find a value of c that satisfies the conclusion of the Integral Mean Value Theorem $f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$

$\int_0^2 3x^2\,dx (= 8)$

أوجد قيمة c التي تحقق نتيجة نظرية القيمة المتوسطة للتكامل $f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$.

$\int_0^2 3x^2\,dx (= 8)$

A. $\frac{2}{\sqrt{3}}$

B. $-\frac{2}{\sqrt{3}}$

C. $\sqrt{3}$

D. 1

Question 16 MOE Exam 2022/2023

16

Q.16: 1 التعرف على مفهوم التكامل المحدود (B2)

Write the given (total) area as an integral or sum of integrals.

The area above the x –axis and below $y = 4x - x^2$.

اكتب (مجمل) المساحة المعطاة في صورة تكامل أو ناتج جمع تكاملات.

المساحة فوق المحور – x وتحت $.y = 4x - x^2$

A $\int_0^4 (4x - x^2)dx$

B $\int_{-4}^0 -(4x - x^2)dx$

C $\int_0^4 -(4x - x^2)dx$

D $\int_0^2 (4x - x^2)dx$

A

Term 2 | Questions from previous MOE exams أسئلة وردت بامتحانات وزارية سابقة | Mathematics G12 Advanced

Question 16 MOE Exam 2021/2022

Write the given (total) area as an integral or sum of integrals.

The area above the x –axis and below $y = 4 - x^2$.

اكتب (مجمل) المساحة المعطاة في صورة تكامل أو ناتج جمع تكاملات.

المساحة فوق المحور – x وتحت $.y = 4 - x^2$

A $\int_0^2 -(4 - x^2)dx$

B $\int_{-2}^2 (4 - x^2)dx$

C $\int_0^2 (4 - x^2)dx$

D $\int_{-2}^2 -(4 - x^2)dx$

B

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Question 17 MOE Exam 2021/2022

Compute the average value of $f(x) = 4x + 3$ on the interval $[0,2]$.

احسب القيمة المتوسطة لـ $f(x) = 4x + 3$ على الفترة $[0,2]$.

A. 7

B. 11

C. 22

D. 14

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Question 18 MOE Exam 2021/2022

Write the expression as a single integral.

$$\int_0^5 f(x)dx - \int_2^5 f(x)dx$$

اكتب التعبير في صورة تكامل منفرد.

$$\int_0^5 f(x)dx - \int_2^5 f(x)dx$$

A. $\int_5^2 f(x)dx$

B. $\int_0^2 f(x)dx$

C. $\int_2^5 f(x)dx$

D. $\int_0^5 f(x)dx$

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Question 19 MOE Exam 2021/2022

Assume that

$\int_1^4 f(x)dx = 5$ and $\int_1^4 g(x)dx = -3$.

Find $\int_1^4 [2f(x) - g(x)]dx$.

فرضًا أنّ

$\int_1^4 g(x)dx = -3$ و $\int_1^4 f(x)dx = 5$

أوجد $\int_1^4 [2f(x) - g(x)]dx$.

A 13

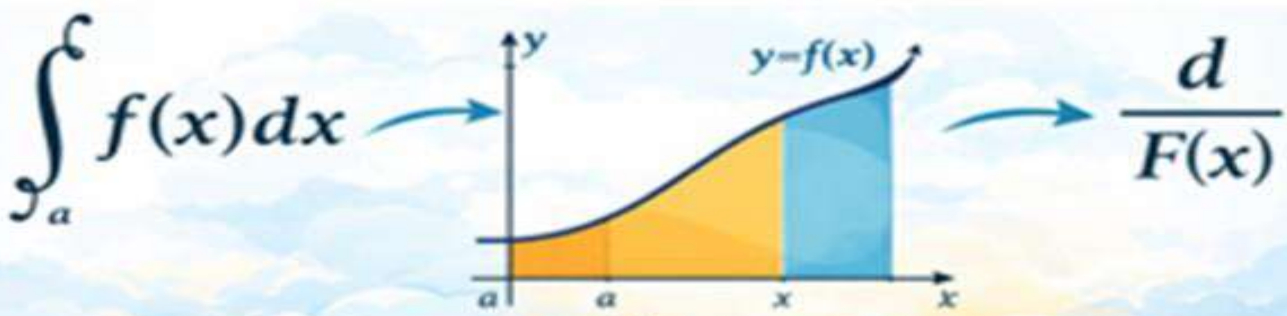
B 2

C 7

D 8

Unit 5: Integration

وحدة 5: التكامل



If f is continuous on $[a, b]$ and $F(x) = \int_a^x f(t)dt$, then $F'(x) = f(x)$, on $[a, b]$

$$\int_a^b f(x)dx = F(x)|_a^b = F(b) - F(a)$$

Lesson 5 : The Fundamental Theorem of Calculus

الدرس الخامس: النظرية الأساسية في التفاضل والتكامل

Grade 12 Advanced

الثاني عشر متقدم

Mr. Abdalla Abouelnaga
0505114830

Student Name (اسم الطالب) ______________________

Class (الصف) ______________________

Unit 5

Integration

Lesson 5

The Fundamental Theorem of Calculus

Objectives

- **Learn the Fundamental Theorem of Calculus (Part I) and use it to compute various definite integrals.**
- **Learn the Fundamental Theorem of Calculus (Part II) and use it to compute derivatives of functions defined as definite integrals.**
- **Write the equation of a tangent line at a given point to a function defined as definite integral.**

Introduction

Fundamental Theorem of Calculus is a theorem that links the concept of differentiating a function with the concept of integrating a function.

Firstly, it provides us with a powerful technique to evaluate definite integrals without using geometric areas and limits of Riemann sums.

Secondly, it is saying that differentiation and integration are in some sense inverse operations.

Theorem 5.1 _ (The Fundamental Theorem of Calculus. Part I)

If f is continuous on $[a, b]$ and F(x) is any antiderivative of $f(x)$, then.

$$\int_a^b f(x)dx = F(b) - F(a)$$

To compute definite integral, find an antiderivative and evaluate it at the two limits of integration, which is much easier than computing limits of Riemann sums.

$$\int_a^b f(x)dx = F(x)|_a^b = F(b) - F(a)$$

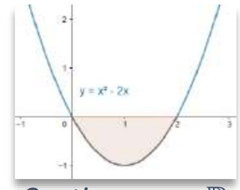
Example: 1

A $If\ F(x) = x\ln x - x + c$ is antiderivative for $f(x)$ find $\int_1^e f(x)dx$

B If $F(x)$ is antiderivative of $f(x) = 3x^2 + 5$, $F(2) = 5$ find $F(1)$

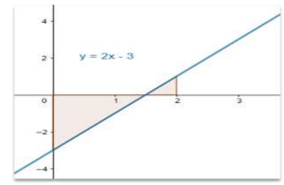
Example: 2 Compute:

a) $\int_0^2 (x^2 - 2x)dx =$



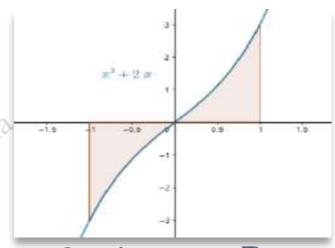
Continuous $x \epsilon \mathbb{R}$

b) $\int_0^2 (2x - 3)dx$ =



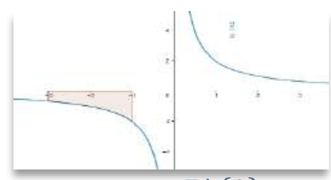
Continuous $x \epsilon \mathbb{R}$

c) $\int_{-1}^{1} (x^3 + 2x)dx =$



Continuous $x \epsilon \mathbb{R}$

d) $\int_{-3}^{-1} \frac{2}{x} dx =$



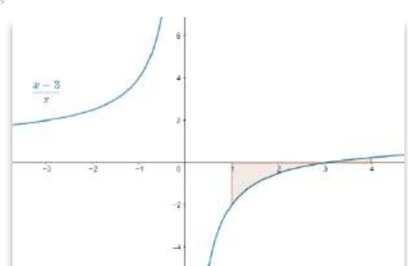
Continuous $x \epsilon \mathbb{R} \backslash \{0\}$

Remember

$$\int \frac{1}{x} dx = ln|x| + c$$

Don't forget the absolute value.

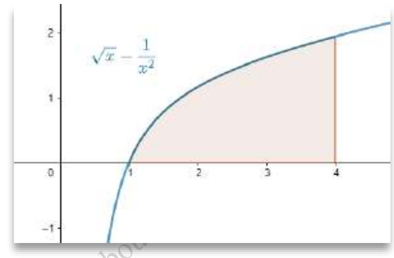
e) $\int_1^4 \left(\frac{t-3}{t}\right) dt =$



Continuous $x \epsilon \mathbb{R} \backslash \{0\}$

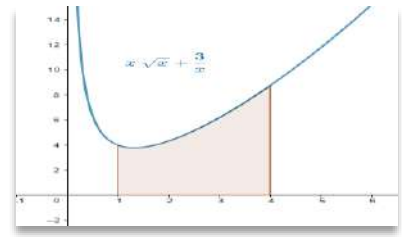
Exercise: 2.1

a) $\int_1^4 \left(\sqrt{x} - \frac{1}{x^2}\right) dx =$



Continuous $x\epsilon(0, \infty)$

b) $\int_1^4 \left(x\sqrt{x} + \frac{3}{x}\right) dx =$



Continuous $x\epsilon(0, \infty)$

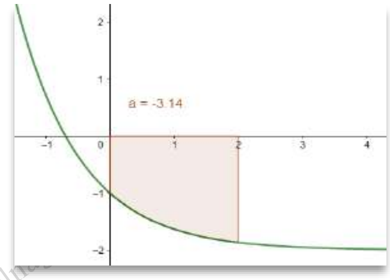
c) $\int_0^{\frac{1}{2}} \left(\frac{3}{\sqrt{1-x^2}}\right) dx =$

d) $\int_0^{\frac{\pi}{4}} sec^2 x \, dx =$

e) $\int_0^{\frac{\pi}{2}} cos^2 x \, dx =$

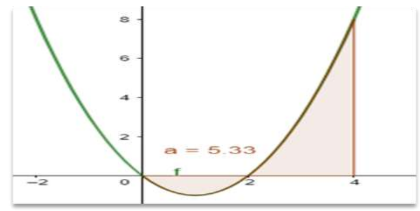
Example: 3

Compute:



$f(x) = e^{-x} - 2$ ***is continuous on*** $[0, 2]$

a) $\displaystyle\int_0^2 \frac{e^{2x} - 2e^{3x}}{e^{3x}}\,dx =$



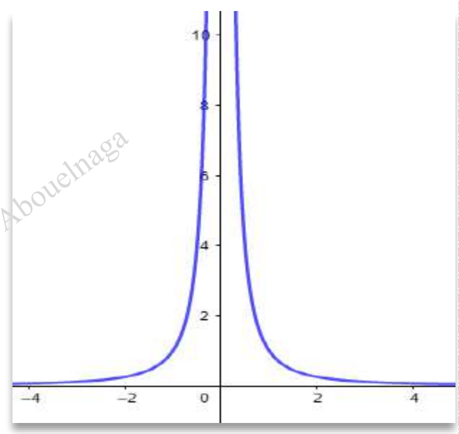
Continuous $t \epsilon \mathbb{R}$

b) $\displaystyle\int_0^4 t(t-2)dt =$

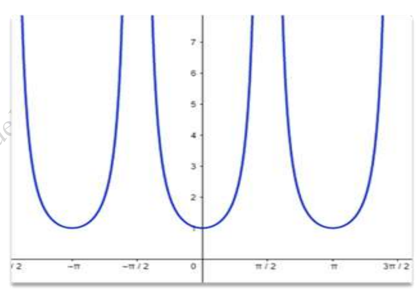
Example 4

Explain how you know the proposed integral value is wrong and find all mistakes.

a) $\displaystyle\int_{-1}^1 \frac{1}{x^2}\,dx = \frac{-1}{x}\Big|_{x=-1}^{x=1} = -1 - (1) = -2$

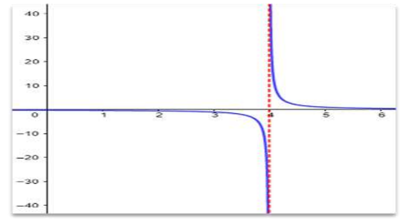


b) $\displaystyle\int_0^{\pi} sec^2 x\,dx = tanx\Big|_{x=0}^{x=\pi} = tan\pi - tan0 = 0$

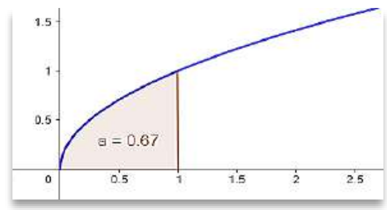


Exercise: 4.1 Identify the integrals to which fundamental theorem of calculus applies; the other integrals are called improper integrals.

$a)\ \displaystyle\int_0^4 \frac{1}{x-4}\,dx$

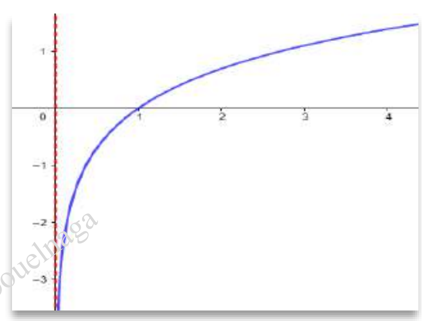


$b)\ \displaystyle\int_0^1 \sqrt{x}\ dx$



Domain: $[0,\infty)$

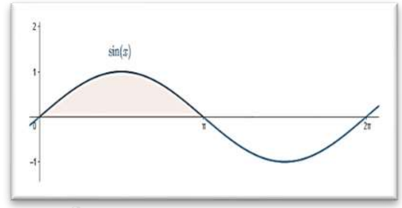
$c)\ \displaystyle\int_0^1 lnx\,dx$



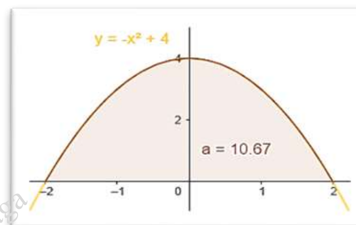
Domain: $(0,\infty)$

Example 5

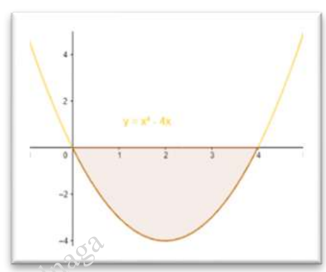
a) Find the area above $\boldsymbol{x-axis}$ and under the curve: $\boldsymbol{f(x)=sinx}$ *on the interval* $[0,\pi]$



b) Find the area above $x-axis$ and below the curve of: $\boldsymbol{y=4-x^2}$

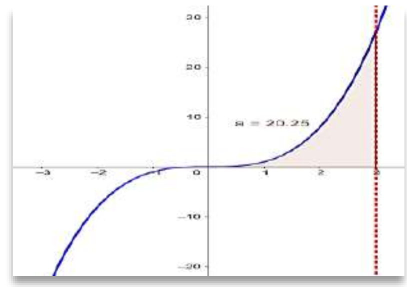


C)Find the area below $\boldsymbol{x-axis}$ and above the curve of: $\boldsymbol{y=x^2-4x}$

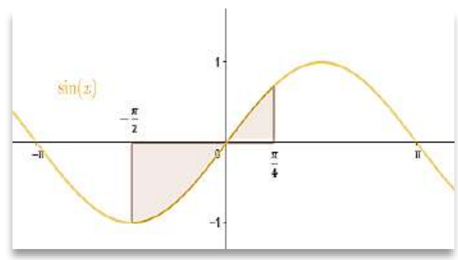


Exercise: 5.1

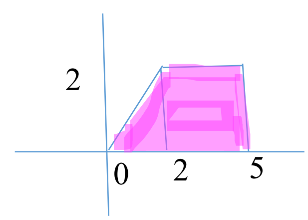
a) Find the area of the region bounded by $\boldsymbol{y = x^3}$, $\boldsymbol{x = 3}$ and the $\boldsymbol{x - axis}$



b) Find the area between $\boldsymbol{y = sinx}$ and the $x - axis$ for $\boldsymbol{-\pi/2 \leq x \leq \pi/4}$



c) Find the area between $f(x) = \begin{cases} x, & x < 2 \\ 2, & x \geq 2 \end{cases}$ $and\ x\ axis\ on\ [0,5]$



Theorem 5.1 _ (The Fundamental Theorem of Calculus. Part II)

If f is continuous on $[a,b]$ and $F(x)=\int_a^x f(t)dt$, then $F'(x)=f(x)$, on $[a,b]$

The antiderivative function F is defined as an integral of the function f

Part II says that every continuous function f has an antiderivative, namely, $\int_a^x f(t)dt$

Three Cases

1) Upper is x — If f is continuous on $[a,b]$ and $F(x)=\int_a^x f(t)dt$, then $F'(x)=f(x)$, on $[a,b]$

2) Upper is a function — If $F(x)=\int_a^{u(x)} f(t)dt$, then $F'(x)=f(u(x))u'(x)$ (chain rule and Fundamental Theorem

3) Upper and lower are variables — If $F(x)=\int_{g(x)}^{u(x)} f(t)dt$, then $F'(x)=f(u(x))u'(x)$ - $f(g(x))g'(x)$

Example 6

Find the derivative $F'(x)$

a) $F(x)=\int_1^x (t^2-2t+3)dt$

b) $F(x)=\int_0^x (t^2-3t+2)\,dt$

c) $F(x)=\int_2^x (t^2-3t-4)\,dt$

d) $F(x)=\int_x^2 sect\;\,dt$

Exercise6.1

a) $F(x)=\int_x^1 t\,e^{2t}\;dt$

b) $F(x)=\int_0^{\frac{\pi}{4}} t\,e^{2t}\;dt$

c) $If\ F(x)=\int_1^x \sqrt{4t^2-1}\;dt\ find\ F(1)\,,F'(1)$

Example 7 Find

a) $\int_0^t (e^{x/2})^2\, dx$

b) $\int_0^t (sin^2 x + cos^2 x)\, dx$

c) $\int_1^x 12t^5 dt$

Using The Chain Rule and Fundamental Theorem of Calculus. Part II

If $g(x) = \int_a^{u(x)} f(t)dt$, then $g'(x) = f\big(u(x)\big)u'(x)$

Or

$$\frac{d}{dx}\int_a^{u(x)} f(t)dt = f\big(u(x)\big)u'(x)$$

Example 8 If $F(x)=\int_2^{x^2} cost\,dt$, compute $F'(x)$

Exercise 8.1 If $f(x)=\int_0^{x^2}\left(e^{-t^2}+1\right)dt$, compute $f'(x)$

Example 9 If $F(x)=\int_{2x}^{x^2}\sqrt{t^2+1}\,dt$, compute $F'(x)$

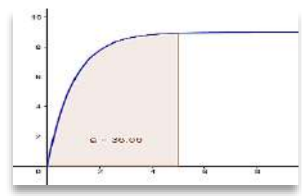
Exercise 9.1 If $f(x)=\int_{e^x}^{2-x} sint^2 dt$, compute $f'(x)$

c) If $f(x) = \int_{2-x}^{xe^x} e^{2t} dt$, compute $f'(x)$

Computing the Distance Fallen by an Object

Example 10

a) Suppose the downward velocity of a sky diver is given by $v(t) = 9(1 - e^{-t}) m/s$ for the first 5 *seconds* of a jump. Compute the distance fallen.



b) Find the position function $s(t)$
if $v(t) = 40 - sint\ ft/sec, s(0) = 2$

Exercise10.1

a) Find the position function $s(t)$ if $v(t) = 10e^{-t} ft/sec, s(0) = 2$

Acceleration, Velocity and Position Functions

Example 11

a) Find the position function $s(t)$ if $a(t) = 4 - t \;\; ft/sec^2$, $v(0) = 8, \;\; s(0) = 0$

Exercise11.1

b) Find the position function $s(t)$ if $a(t) = 16 - t^2 ft/sec^2$, $v(0) = 0, \;\; s(0) = 30$

Rate of Change and Total Change of Volume of a Tank

Example 12

Suppose that water flows <u>in</u> and <u>out</u> of a storage tank. The <u>net</u> rate of change (that is the rate in minus the rate out) of water is $f(t) = 20(t^2 - 1)$ liter per minute.

a) For $0 \leq t \leq 3$, determine when the water level is increasing and when the water level is decreasing.

b) If the tank has 200 liters of water at time $t = 0$, determine how many litters are in the tank at time $t = 3$ minutes.

Exercise12.1

Suppose that the rate of change of water in a storage tank is $f(t) = 10\sin t$ gallons per minute.

a) For $0 \leq t \leq 2\pi$, determine when the water level is increasing and when the water level is decreasing.

b) If the tank has 100 gallons of water at time $t = 0$, determine how many gallons are in the tank at time $t = \pi$.

Finding a Tangent Line for a Function Defined as an Integral

Example 13

a) For the function $\boldsymbol{F(x) = \int_4^{x^2} ln(t^3+4)\,dt}$, find the equation of the tangent line at $x = 2$

b) For the function $y = \int_0^x sin\sqrt{t^2+\pi^2}dt$, find the equation of the tangent line at $x = 0$

Exercise13.1

For the function $\boldsymbol{y = \int_0^x e^{-t^2+1}dt}$, find the equation of the tangent line at $x = 0$

Question 8 MOE Exam 2023/2024

Q.8: Learn the Fundamental Theorem of Calculus (Part I) and use it to compute various definite integrals

Evaluate $\int_0^{\pi/4}(\sin x - \cos x)dx$. أوجد قيمة $\int_0^{\pi/4}(\sin x - \cos x)dx$.

A	B	C	D
$\sqrt{2}-1$	$\sqrt{2}+1$	$-\sqrt{2}-1$	$-\sqrt{2}+1$

Question 5 MOE Exam 2023/2024

Question 5	Learn the Fundamental Theorem of Calculus (Part II) and use it to compute derivatives of functions defined as definite integrals التعرف على النظرية الأساسية الثانية للتفاضل والتكامل وتطبيقها على دوال معرفة كتكاملات محدودة لإيجاد مشتقاتها	السؤال 5

If إذا كانت $f(x) = \int_{3x}^{\sin x}(t^2+4)\,dt.$ Compute $f'(x)$ أوجد

Solution

انتهت أسئلة الامتحان

Question 13 MOE Exam 2022/2023

13

Q.13: 1 التعرف على النظرية الأساسية الأولى للتفاضل والتكامل وتطبيقها على دوال متنوعة لإيجاد تكاملات محدودة Mark(s): 4/4

(B2)

Evaluate.

$$\int \left(\frac{e^{2x} - 2e^{3x}}{e^{3x}}\right) dx$$

أوجد قيمة.

$$\int \left(\frac{e^{2x} - 2e^{3x}}{e^{3x}}\right) dx$$

A $-\frac{1}{e^x} - 2x + c$

B $\frac{1}{e^x} - x + c$

C $-\frac{1}{e^x} - 2xe^x + c$

D $e^x - 2x + c$

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Question 21 MOE Exam 2021/2022

If $f(x) = \int_x^{x^2} \sin 3t\, dt$, compute $f'(x)$.

إذا كانت $f(x) = \int_x^{x^2} \sin 3t\, dt$ ، احسب $f'(x)$.

A $f'(x) = 2x\sin 3x^2 + \sin 3x$

B $f'(x) = 2x\sin 3x^2 - \sin 3x$

C $f'(x) = \sin 3x^2 - \sin 3x$

D $f'(x) = \sin 3x - 2x\sin 3x^2$

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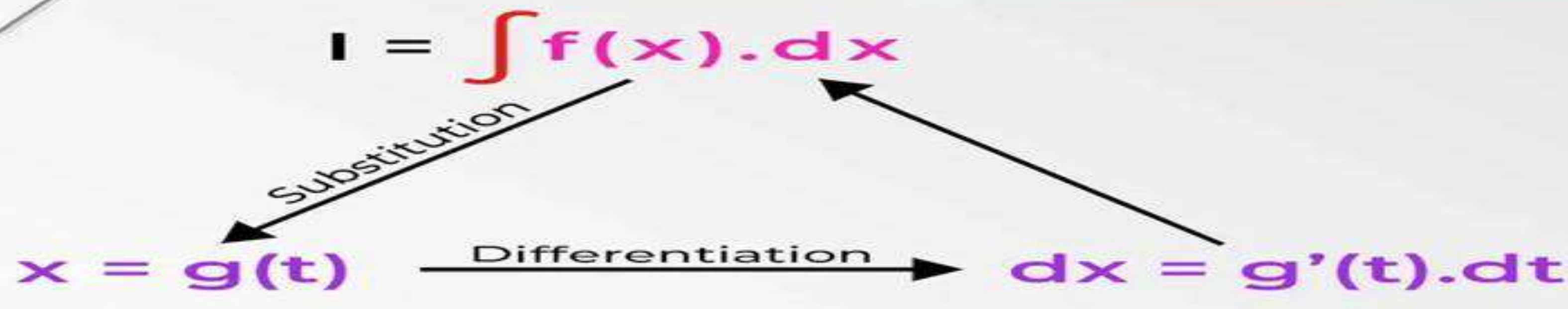
Unit 5: Integration

وحدة 5: التكامل

Integration by Substitution



$$I = \int f[g(t)].g'(t).dt$$

Lesson 6 : Integration by Substitution

الدرس السادس: التكامل بالتعويض

Grade 12 Advanced

الثاني عشر متقدم

Mr. Abdalla Abouelnaga

Mr. Abdalla Abouelnaga
0505114830

Student Name (اسم الطالب:) ____________________

Class (الصف) ____________________

Unit 5

Integration

Lesson 6

Integration by Substitution

Objectives

- Compute integrals using substitution.

Introduction

If you have the indefinite integral

$$\int (2x+3)e^{x^2+3x}dx$$

It seems a difficult integral

Here we have $\frac{d}{dx}[x^2+3x] = 2x+3$ *Part of the integral is the derivative of another part*

To integrate such functions, we are going to use U Substitution ⇨ *Integration by Substitution*

Which is unwinding of the chain rule in differentiation

if we have $\frac{d}{dx}\left[e^{x^2+3x}\right] = \frac{d}{dx}[x^2+3x]\cdot e^{x^2+3x}$

Using chain rule $= (2x+3)\cdot e^{x^2+3x}$

So, to integrate $\int (2x+3)e^{x^2+3x}dx = e^{x^2+3x} + c$

Integration by Substitution

Integration by substitution consists of the following steps:

- Choose a new variable u: a common choice is the innermost expression or "inside" term of a composition of functions.
- Compute: $du = \frac{du}{dx}dx$
- Replace all terms in the original integrand with expression involving u and du
- Evaluate the resulting (u) integral. If you still can't evaluate the integral, you may need to try a different choice of u
- Replace each occurrence of u in the antiderivative with the corresponding expression in x.

Example: 1 Evaluate:

a) $\int (x^3+5)^{100}(3x^2)dx$

b) $\int x\cos x^2 dx$

Note: *When choosing* u

- *Choose a part of the function that you have its derivative.*
- *Choose a troublesome term.*

c) $\int (3\tan x+4)^5 \sec^2 x dx$

Exercise: 1.1 Evaluate:

a) $\int \frac{(tan^{-1}x)^2}{1+x^2}dx$

b) $\int \frac{tan^{-1}2x}{1+4x^2}dx$

c) $\int \frac{sin\sqrt{x}}{\sqrt{x}}dx$

Example: 2 Evaluate:

$a) \int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx$

$b) \int \frac{cos(1/x)}{x^2} dx$

Exercise: 2.1

$a) \int \frac{\sqrt{lnx}}{x} dx$

$b \int tanxln(cosx)dx$

Theorem 6.1

For any continuous function $\boldsymbol{f(x)}$

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c$$

On any interval in which $\boldsymbol{f(x) \neq 0}$

Example: 3 Evaluate:

$a)\ \int \frac{x^2}{x^3+5}\,dx$

$b)\ \int \tan x\,dx$

Exercise: 3.1 Evaluate:

$a)\ \int \frac{x}{2x^2+1}\,dx$

$b \int \frac{1}{1+e^x} dx$

Example: 4 Evaluate:

a) $\int \frac{x}{\sqrt{1-x^4}} dx$

b) $\int \frac{x^3}{\sqrt{1-x^4}} dx$

c) $\int \frac{x^2}{1+x^6} dx$

d) $\int \frac{x^5}{1+x^6} dx$

Exercise: 4.1 Evaluate:

a) $\int \frac{1+x}{1+x^2} dx$

b) $\int \frac{1+x}{1-x^2} dx$

Example: 5 Evaluate:

$a)\ \int x\sqrt{2-x}\,dx$

$b)\ \int \frac{2t+3}{t+7}\,dt$

$c)\ \int \frac{t^2}{\sqrt[3]{t+3}}\,dt$

Exercise: 5.1 Evaluate:

$a) \int \frac{1}{1+\sqrt{x}}\,dx$

$b) \int \frac{1}{x-x^{2/3}}\,dx$

Substitution in Definite Integral

$$\int_a^b f\big(u(x)\big)u'(x)dx = \int_{u(a)}^{u(b)} f(u)du$$

The limits of integration change from

$$x = a \quad \Rightarrow \quad u: u(a)$$

$$x = b \quad \Rightarrow \quad u: u(b)$$

Example: 6 Evaluate:

a) $\int_{1}^{2} x^3\sqrt{x^4+5}\,dx$

b) $\int_{-1}^{1} \frac{t}{(t^2+1)^2}\,dt$

c) $\int_{0}^{15} te^{-t^2/2}dt$

Exercise: 6.1

a) $\int_0^2 \frac{e^x}{1+e^{2x}}dx$

b) $\int_0^2 \frac{e^x}{1+e^x}dx$

Example: 7

Make the indicated substitution for an unspecified function.

a) $u = x^2$ for $\int_0^2 xf(x^2)dx$

b) $u = x^3$ for $\int_1^2 x^2 f(x^3)dx$

Exercise: 7.1

a) Find $\int_0^{\pi/2} \cos x . f(\sin x)dx$ if $\int_0^1 f(x)dx = 2$

b) Find $\int_1^4 \frac{f(\sqrt{x})}{\sqrt{x}} dx$ if $\int_1^2 f(x)dx = 3$

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Question 11 MOE Exam 2022/2023

11

Q.11: 1 استخدام طريقة التكامل بالتعويض لإيجاد تكاملات (B2)

Evaluate the indicated integral.

أوجد قيمة التكامل غير المحدود.

$$\int \tan 2x \, dx$$

A. $-\cot 2x$

B. $\frac{\sin 2x}{\cos 2x} + c$

C. $-\frac{1}{2}\ln|\sin 2x| + c$

D. $-\frac{1}{2}\ln|\cos 2x| + c$

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Question 12 MOE Exam 2022/2023

12

Q.12: 1 استخدام طريقة التكامل بالتعويض لإيجاد تكاملات (B2)

Evaluate the indicated integral.

أوجد قيمة التكامل غير المحدود.

$$\int x^3\sqrt{x^4+3}\, dx$$

A. $\frac{1}{2}(x^4+3)^{\frac{3}{2}} + c$

B. $\frac{1}{6}x^4(x^5+3)^{\frac{3}{2}} + c$

C. $\frac{1}{6}(x^4+3)^{\frac{1}{2}} + c$

D. $\frac{1}{6}(x^4+3)^{\frac{3}{2}} + c$

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