هيكل مراجعة متقدم Reveal



تم تحميل هذا الملف من موقع المناهج الإماراتية	
موقع المناهج ⇦ المناهج الإماراتية ⇦ الصف الثاني عشر المتقدم ⇦ رياضيات ⇦ الفصل الأول ⇦ ملفات متنوعة ⇦ الملف	
تاريخ إضافة الملف على موقع المناهج: 2026-11-24 00:10:25	
ملفات ا كتب للمعلم ا كتب للطالب ا اختبارات الكترونية ا اختبارات ا حلول ا عروض بوربوينت ا أوراق عمل منهج انجليزي ا ملخصات وتقارير ا مذكرات وبنوك ا الامتحان النهائي ا للمدرس	المزيد من مادة رياضيات:
إعداد: عبدالرحمن بدر	

التواصل الاجتماعي بحسب الصف الثاني عشر المتقدم					
صفحة المناهج الإماراتية على فيسبوك					
المواد على تلغرام	التربية الاسلامية	اللغة العربية	اللغة الانجليزية	الرياضيات	

المزيد من الملفات بحسب الصف الثاني عشر المتقدم والمادة رياضيات في الفصل الأول	
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تطبيق المناهج على Store App (آبل)	تطبيق المناهج على Play Google (أندرويد)

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هيكل الماث ثانى عشر متقدم - الفصل الاول



إعداد المهندس عبدالرحمن بدر
+201002507441

Q1	Estimate an arc length of a given function تقدير طول القوس على منحنى دالة معطاة	7-12	68 70

7 **Estimate the length of the curve $y = \cos x$ on $0 \le x \le \frac{\pi}{2}$ using $n = 2$ line segments.**

قدر طول القوس على المنحنى $y = \cos x$ في الفترة $0 \le x \le \frac{\pi}{2}$ باستخدام $n = 2$ مقطعي خط مستقيم.

(A) Length ≈ 2.2432

(B) Length ≈ 3.4011

(C) Length ≈ 1.0121

(D) Length ≈ 1.8950

8 **Estimate the arc length of the curve $y = (x-1)^2$ on the interval $0 \le x \le 3$ using $n = 2$ line segments.**

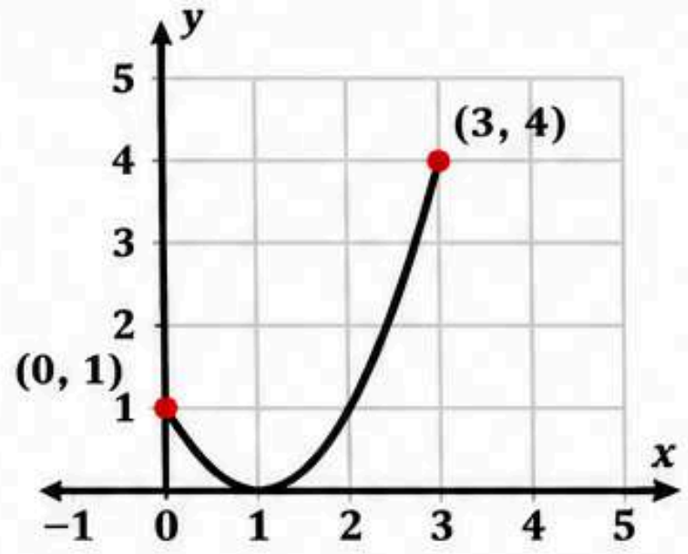
قدر طول القوس على المنحنى $y = (x-1)^2$ في الفترة $0 \le x \le 3$ باستخدام $n = 2$ مقطعي خط مستقيم.

(A) Length ≈ 4.0359

(B) Length ≈ 2.8125

(C) Length ≈ 32.6717

(D) Length ≈ 5.7159



9 **Estimate the length of the curve $y = \sqrt{x+1}$ on $0 \le x \le 3$ using $n = 2$ line segments.**

قدر طول القوس على المنحنى $y = \sqrt{x+1}$ في الفترة $0 \le x \le 3$ باستخدام $n = 2$ مقطعي خط مستقيم.

(A) Length ≈ 2.821

(B) Length ≈ 3.166

(C) Length ≈ 5.021

(D) Length ≈ 4.06

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10 Estimate the length of the curve $y = \frac{1}{x}$ on $1 \le x \le 2$ using $n = 3$ line segments.

قدر طول القوس على المنحنى $y = \frac{1}{x}$ في الفترة $1 \le x \le 2$ باستخدام $n = 3$ مقطعي خط مستقيم.

a) Length ≈ 0.8512

b) Length ≈ 1.1302

c) Length ≈ 1.861

d) Length ≈ 3.423

11 Estimate the length of the curve $y = x^2 + 1$ on $-2 \le x \le 2$ using $n = 2$ line segments.

قدر طول القوس على المنحنى $x^2 + 1$ في الفترة $-2 \le x \le 2$ باستخدام $n = 2$ مقطعي خط مستقيم.

a) Length ≈ 8.944

b) Length ≈ 9.495

c) Length ≈ 9.847

d) Length ≈ 12.5232

12 Estimate the length of the curve $y = x^3 + 2$ on $-1 \le x \le 1$ using $n = 3$ line segments.

قدر طول القوس على المنحنى $x^3 + 2$ في الفترة $-1 \le x \le 1$ باستخدام $n = 3$ مقطعي خط مستقيم.

a) Length ≈ 1.462

b) Length ≈ 2.124

c) Length ≈ 12.542

d) Length ≈ 3.013

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Q2	Find a limit algebraically or graphically, if it exists	(9-11)	85

2) Use the shown graph represented $f(x)$ to answer the following question

A) 2

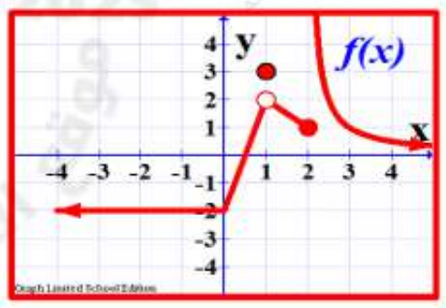
B) 3

C) غير موجودة DNE

D) 1

2) استخدم الرسم المجاور للدالة في الإجابة عن السؤال الآتي $f(x)$

$\lim_{x\to 1} f(x)$



2) Use the shown graph represented $f(x)$ to answer the following question

A) 3

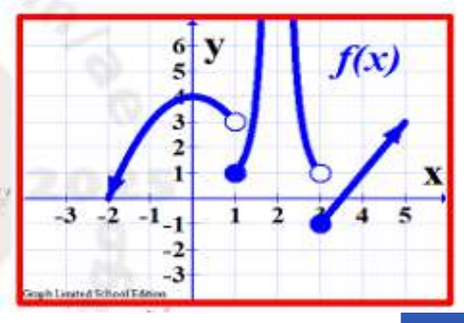
B) 2

C) DNE غير موجودة

D) ∞

2) استخدم الرسم المجاور للدالة في الإجابة عن السؤال الآتي $f(x)$

$\lim_{x\to 2} f(x)$



2) Use the shown graph represented $f(x)$ to answer the following question

A) -2

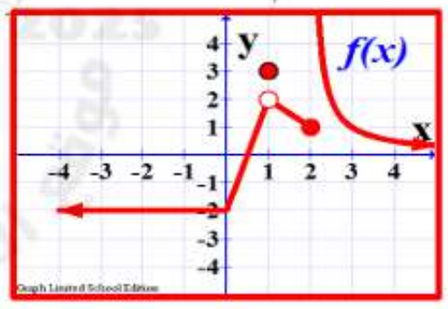
B) 1

C) 0

D) ∞

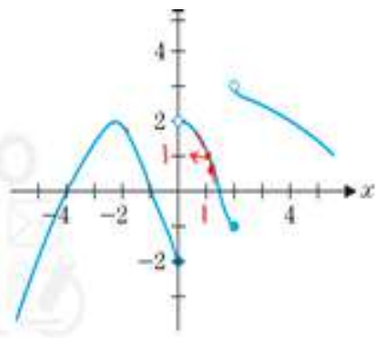
2) استخدم الرسم المجاور للدالة في الإجابة عن السؤال الآتي $f(x)$

$\lim_{x\to 2^-} f(x)$



أوجد قيمة $\lim_{x\to 0} \frac{1-\cos^2 x}{x^2}$ إن وجدت.

A	غير موجودة	5
B	2	
C	1	
D	0	



ي مما يلي:

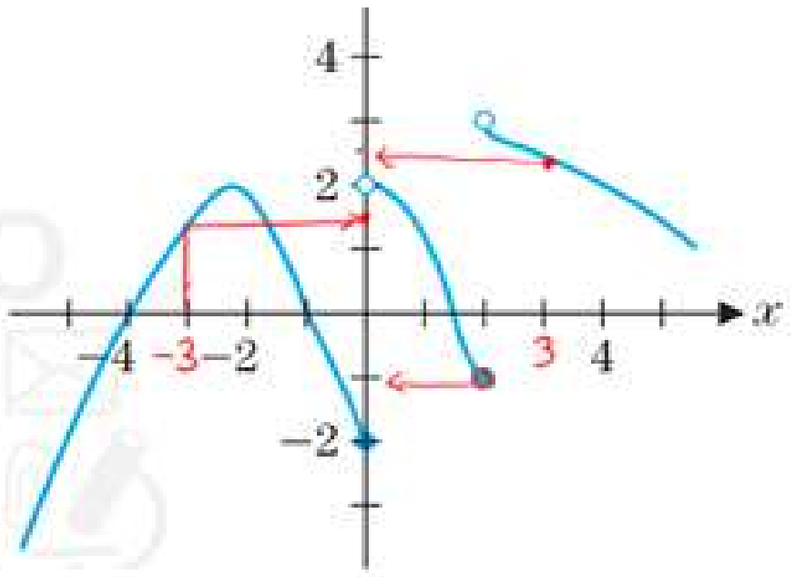
أوجد قيمة $\lim_{x\to 2}\left(\frac{1}{x-2}-\frac{4}{x^2-4}\right)$ إذا وُجدت.

A	$\frac{1}{2}$	4
B	غير موجودة	
C	2	
D	$\frac{1}{4}$	

7. (a) $\lim_{x\to 0^-} f(x) = -2$ (b) $\lim_{x\to 0^+} f(x) = 2$ (c) $\lim_{x\to 0} f(x)$ $0^- \neq 0^+$ = غير موجودة

(d) $\lim_{x\to -2^-} f(x) = 2$ (e) $\lim_{x\to -2^+} f(x) = 2$ (f) $\lim_{x\to -2} f(x)$ $-2^- = -2^+ = 2$

(g) $\lim_{x\to -1} f(x) = 0$ (h) $\lim_{x\to 1^-} f(x) = 1$



8. (a) $\lim_{x\to 1^-} f(x) = 1$ (b) $\lim_{x\to 1^+} f(x) = 1$ (c) $\lim_{x\to 1} f(x) = 1$

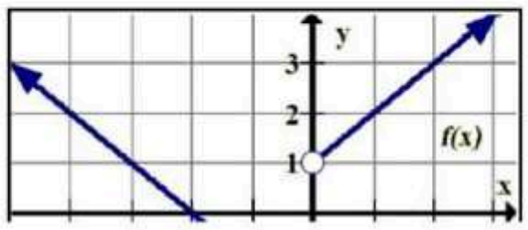
(d) $\lim_{x\to 2^-} f(x) = -1$ (e) $\lim_{x\to 2^+} f(x) = 3$ (f) $\lim_{x\to 2} f(x)$ $2^- = -1$ $2^+ = 3$ غير موجودة

(g) $\lim_{x\to 3^-} f(x) = 2.5$ (h) $\lim_{x\to -3} f(x) = 1.5$

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Use the graph shown below to determine $\lim_{x\to 0^+} f(x)$.

استخدم التمثيل البياني أدناه لتحديد $\lim_{x\to 0^+} f(x)$.



a. -2

b. 0

c. 1

d. غير موجودة

does not exist

Evaluate $\lim_{x\to 0}\left(\frac{\frac{1}{x+2}-\frac{1}{2}}{x}\right)$, if it exists.

أوجد قيمة $\lim_{x\to 0}\left(\frac{\frac{1}{x+2}-\frac{1}{2}}{x}\right)$، إن وُجدت.

a. $-\frac{1}{4}$

b. 0

c. $\frac{1}{4}$

d. غير موجودة

does not exist

Evaluate $\lim_{x\to 0}\frac{1-\cos x}{x\sin x}$, if it exists.

أوجد قيمة $\lim_{x\to 0}\frac{1-\cos x}{x\sin x}$، إن وُجدت.

a. 0

b. $\frac{1}{2}$

c. 2

d. غير موجودة

does not exist

21. $\lim_{x\to 2} f(x)$, حيث $f(x)=\begin{cases} 2x & , \quad x<2 \\ x^2 & , \quad x\geq 2 \end{cases}$

A.	4	C.	-2
B.	2	D.	4

22. $\lim_{x\to -1} f(x)$, حيث $f(x)=\begin{cases} x^2+1 & , \quad x<-1 \\ 3x+1 & , \quad x\geq -1 \end{cases}$

A.	Does not exist	C.	1
B.	-1	D.	$\frac{2}{3}$

1 $\lim_{x\to 0}(x^2-3x+1)$

2 $\lim_{x\to 0}\sqrt{2x+1}$

3 $\lim_{x\to 1}\cos^{-1}(x^2)$

4 $\lim_{x\to 3}\frac{x-5}{x^2-4}$

5 $\lim_{x\to 3}\frac{x^2-x-6}{x-3}$

6 $\lim_{x\to 2}\frac{x^2+x-2}{x^2-4}$

7 $\lim_{x\to 2}\frac{x^2-x-2}{x^2-3x+2}$

8 $\lim_{x\to 1}\frac{x^3-1}{x^2+2x-3}$

9 $\lim_{x\to 0}\frac{\sin x}{\tan x}$

10 $\lim_{x\to 0}\frac{|\sin x|}{x}$

11 $\lim_{x\to 0}\frac{xe^{2x}+1}{x^2+x}$

12 $\lim_{x\to 0} x^2\csc^2 x$

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If $f(x) = x^4 - 5x$ then $\lim_{x\to 2} \frac{f'(x)-f'(2)}{x-2}$ equals:

إذا كانت $f(x) = x^4 - 5x$ فإن $\lim_{x\to 2} \frac{f'(x)-f'(2)}{x-2}$ تساوي:

a. 6

b. 27

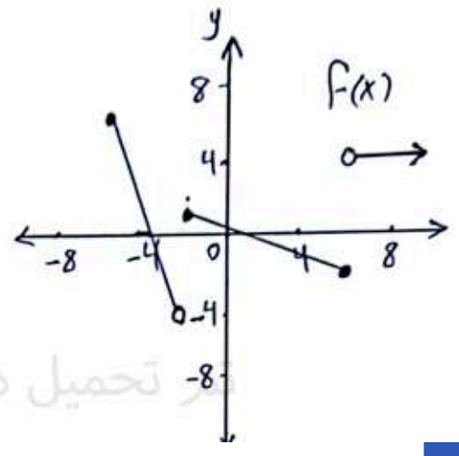
c. 38

d. 48

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(23) استخدم التمثيل البياني لتحديد $\lim_{x\to -2} f(x)$ إن وجدت

Use the graph to determine $\lim_{x\to -2} f(x)$ if it exists



(a) غير موجودة
does not exist

(b) 0

(c) -4

(d) 1

تم تحميل هذا الملف من

(18) إذا كانت $\lim_{x\to 0} \frac{\sin x}{x} = 1$ فأوجد قيمة $\lim_{x\to 0} \frac{\cos^2 x - 1}{2x^2}$

Given that $\lim_{x\to 0} \frac{\sin x}{x} = 1$

evaluate $\lim_{x\to 0} \frac{\cos^2 x - 1}{2x^2}$

(a) $\frac{1}{2}$

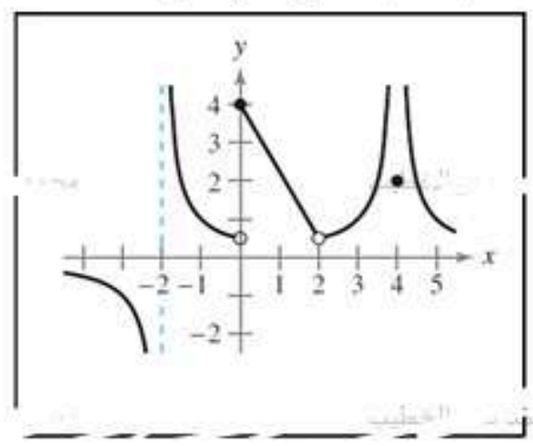
(b) 2

(c) $\frac{-1}{2}$

(d) 1

موقع المناهج الإماراتية
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استخدم الرسم البياني التالي الذي يمثل بيان الدالة $f(x)$ في الإجابة عن الأسئلة التالية :



(1) $f(0)$

(2) $\lim_{x\to 0^+} f(x)$

(3) $\lim_{x\to 0^-} f(x)$

(4) $\lim_{x\to 0} f(x)$

(5) $f(2)$

(6) $\lim_{x\to 2^+} f(x)$

(7) $\lim_{x\to 2^-} f(x)$

(8) $\lim_{x\to 2} f(x)$

(9) $f(4)$

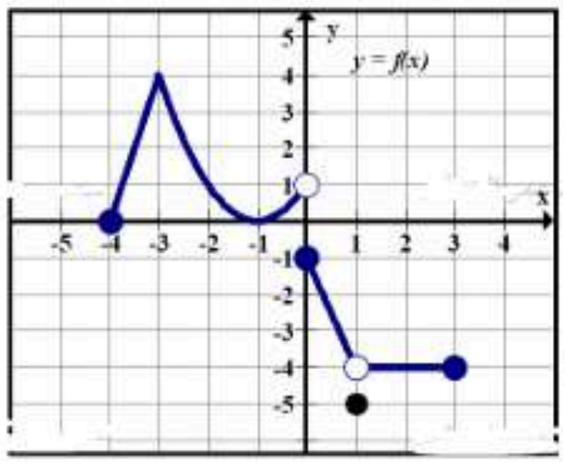
(10) $\lim_{x\to 4^-} f(x)$

(11) $\lim_{x\to 4^+} f(x)$

(12) $\lim_{x\to 4} f(x)$

(13) $\lim_{x\to -2} f(x)$

استخدم الرسم البياني التالي الذي يمثل بيان الدالة $f(x)$ في الإجابة عن الأسئلة التالية :

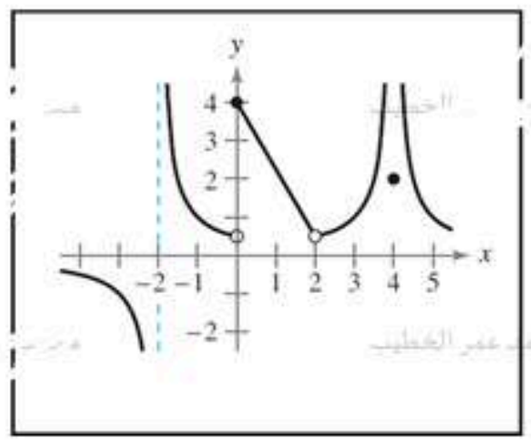


(1) $f(0)$

(2) $\lim_{x\to 0^+} f(x)$

(3) $\lim_{x\to 0^-} f(x)$

(4) $\lim_{x\to 0} f(x)$

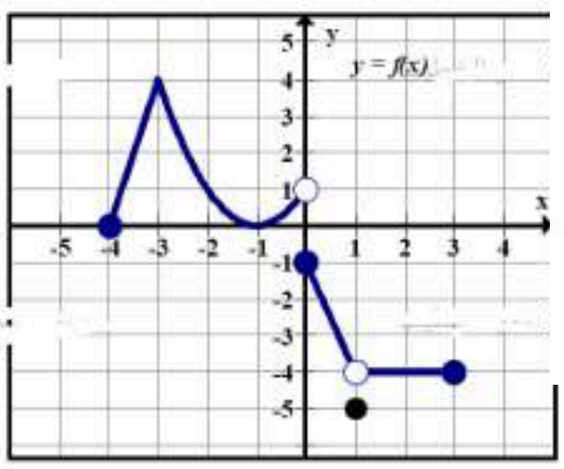


(5) $f(1)$

(6) $\lim_{x\to 1^+} f(x)$

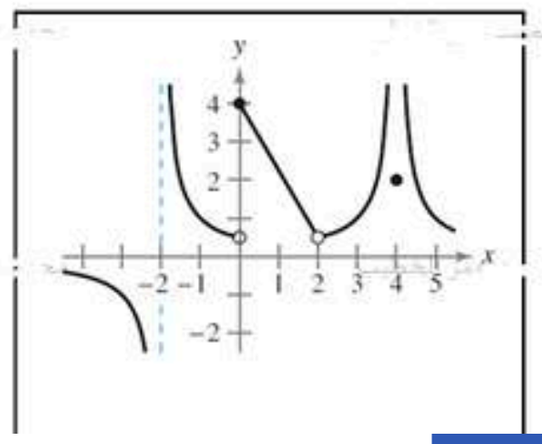
(7) $\lim_{x\to 1^-} f(x)$

(8) $\lim_{x\to 1} f(x)$



(9) $\lim_{x\to 2} f(x)$

(10) $\lim_{x\to -3} f(x)$



23. $\lim_{x\to -1} f(x)$, حيث $f(x) = \begin{cases} 2x+1 & , \quad x < -1 \\ 3 & , \quad -1 < x < 1 \\ 2x+1 & , \quad x > 1 \end{cases}$

A.	Does not exist	C.	1
B.	-1	D.	0

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Question

In exercises 1–28, evaluate the indicated limit, if it exists.
Assume that $\lim_{x\to 0}\frac{\sin x}{x}=1$.

21 $\lim_{x\to 2} f(x)$, where $f(x)=\begin{cases}2x, & x<2\\ x^2, & x\geq 2\end{cases}$

A. 4 **B.** $\frac{5}{2}$ **C.** -3 **D.** 12

Solution

1 Find the one-sided limits:

- Left-hand limit: $\lim_{x\to 2^-} f(x)=\lim_{x\to 2^-} 2x=2(2)=4.$
- Right-hand limit: $\lim_{x\to 2^+} f(x)=\lim_{x\to 2^+} x^2=2^2=4.$

2 Compare the limits:

$$\lim_{x\to 2^-} f(x)=\lim_{x\to 2^+} f(x)=4.$$

3 Conclusion:

The two one-sided limits are equal, so the limit exists and

$$\lim_{x\to 2} f(x)=4.$$

Answer: A. 4

Question

In exercises 1–28, evaluate the indicated limit, if it exists.
Assume that $\lim_{x\to 0}\frac{\sin x}{x}=1$.

22 $\lim_{x\to -1} f(x)$, where $f(x)=\begin{cases}x^2+1, & x<-1\\ 3x+1, & x\geq -1\end{cases}$

A. *does not exist* **B.** $\frac{5}{2}$ **C.** -3 **D.** 12

Solution

1 Find the one-sided limits:

- Left-hand limit: $\lim_{x\to -1^-} f(x)=\lim_{x\to -1^-}(x^2+1)=(-1)^2+1=2.$
- Right-hand limit: $\lim_{x\to -1^+} f(x)=\lim_{x\to -1^+}(3x+1)=3(-1)+1=-2.$

2 Compare the limits:

$$\lim_{x\to -1^-} f(x)=2 \qquad \lim_{x\to -1^+} f(x)=-2.$$

3 Conclusion:

Since the two one-sided limits are not equal, the limit $\lim_{x\to -1} f(x)$ *does not exist.*

Answer: A. *does not exist*

Question

In exercises 1–28, evaluate the indicated limit, if it exists.
Assume that $\lim_{x\to 0}\frac{\sin x}{x}=1$.

23 $\lim_{x\to -1} f(x)$, where $f(x)=\begin{cases}2x+1, & x<-1\\ 3, & -1<x<1\\ 2x+1, & x>1\end{cases}$

A. *does not exist* **B.** $\frac{5}{2}$ **C.** -3 **D.** 12

Solution

1 Find the one-sided limits:

- Left-hand limit: $\lim_{x\to -1^-} f(x)=\lim_{x\to -1^-}(2x+1)=2(-1)+1=-1.$
- Right-hand limit: $\lim_{x\to -1^+} f(x)=\lim_{x\to -1^+} 3=3.$

2 Compare the limits: $\lim_{x\to -1^-} f(x)=-1\neq 3=\lim_{x\to -1^+} f(x).$

3 Conclusion:

The two one-sided limits are not equal, so the limit does not exist. **Answer: A. *does not exist***

Question

In exercises 1–28, evaluate the indicated limit, if it exists.
Assume that $\lim_{x\to 0}\frac{\sin x}{x}=1$.

24 $\lim_{x\to 1} f(x)$, where $f(x)=\begin{cases}2x+1, & x<-1\\ 3, & -1<x<1\\ 2x+1, & x>1\end{cases}$

A. 3 **B.** $\frac{7}{2}$ **C.** -3 **D.** 12

Solution

1 Find the one-sided limits:

- Left-hand limit: $\lim_{x\to 1^-} f(x)=\lim_{x\to 1^-} 3=3.$
- Right-hand limit: $\lim_{x\to 1^+} f(x)=\lim_{x\to 1^+}(2x+1)=2(1)+1=3.$

2 Compare the limits: $\lim_{x\to 1^-} f(x)=3=\lim_{x\to 1^+} f(x).$

3 Conclusion:

The two one-sided limits are equal, so the limit exists and $\lim_{x\to 1} f(x)=3.$ **Answer: A. 3**

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1 The Squeeze Theorem

Suppose that

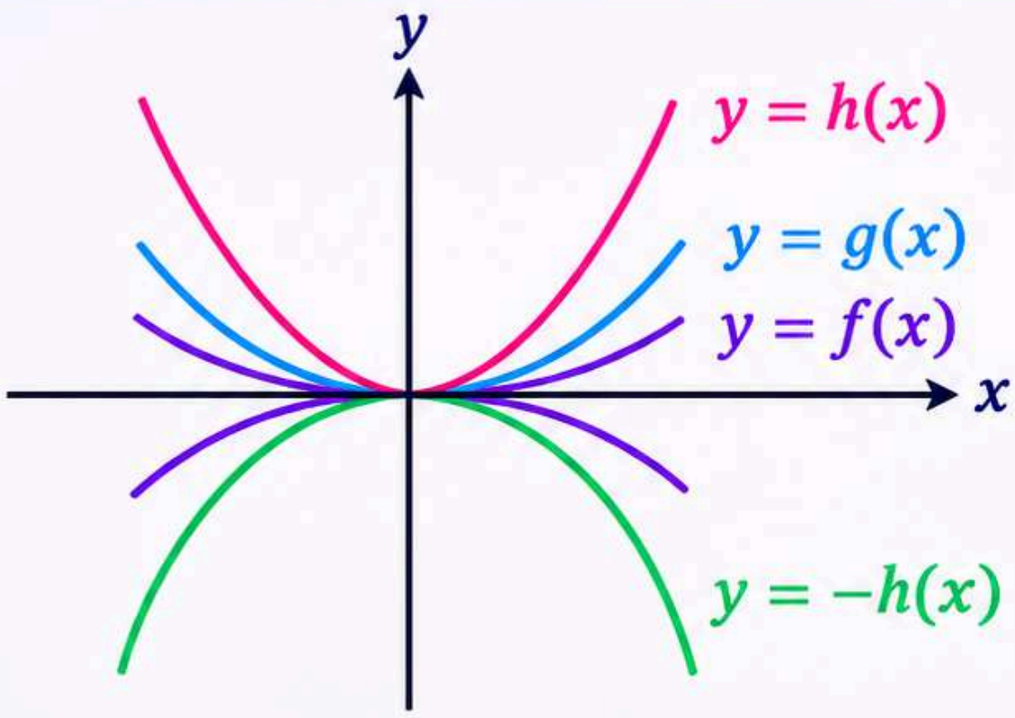
$$f(x) \le g(x) \le h(x)$$

for all x in some interval (c, d), except possibly at the point $a \in (c, d)$, and that

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L,$$

for some number L. Then, it follows that

$$\lim_{x \to a} g(x) = L.$$



The middle curve is squeezed between $y = x^2$ and $y = -x^2$.

4 If $|f(x) - 5| \le x^2$ Find $\lim_{x \to 0} f(x)$

A) 5	C) 0
B) -5	D) *DNE*

Solution

1. From $|f(x) - 5| \le x^2$, we have
$$-x^2 \le f(x) - 5 \le x^2$$
2. Add 5 to all sides:
$$5 - x^2 \le f(x) \le 5 + x^2$$
3. Take the limit as $x \to 0$:
   - $\lim_{x \to 0} (5 - x^2) = 5$
   - $\lim_{x \to 0} (5 + x^2) = 5$
4. By the Squeeze Theorem,
$$\lim_{x \to 0} f(x) = 5$$

2 Example 3.8

Determine the value of $\lim_{x \to 0} \left[x^2 \cos\left(\frac{1}{x}\right)\right]$.

Solution

1. Since $-1 \le \cos\left(\frac{1}{x}\right) \le 1$ for $x \ne 0$, multiplying by $x^2 \ge 0$ gives
$$-x^2 \le x^2 \cos\left(\frac{1}{x}\right) \le x^2.$$
2. As $x \to 0$, both $-x^2$ and x^2 approach 0.
3. By the Squeeze Theorem,
$$\lim_{x \to 0} x^2 \cos\left(\frac{1}{x}\right) = 0.$$

Therefore, the limit is 0.

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29. Use numerical and graphical evidence to conjecture the value of $\lim_{x\to 0} x^2 \sin(1/x)$. Use the Squeeze Theorem to prove that you are correct: identify the functions f and h, show graphically that $f(x) \le x^2 \sin(1/x) \le h(x)$ and justify $\lim_{x\to 0} f(x) = \lim_{x\to 0} h(x)$.

$$f(x) \le x^2 \sin\left(\frac{1}{x}\right) \le h(x)$$

$$-1 \le \sin\left(\frac{1}{x}\right) \le 1$$

x^2

$$-x^2 \le x^2 \sin\left(\frac{1}{x}\right) \le x^2$$

$$\lim_{x\to 0} -x^2 \le \lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) \le \lim_{x\to 0} x^2$$

$$0 \le \lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) \le 0$$

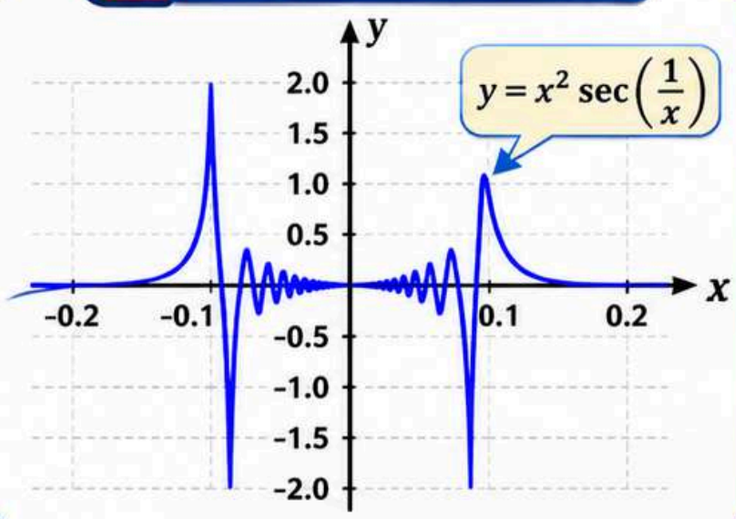
$$\lim_{x\to 0} f(x) = \lim_{x\to 0} h(x) = \lim_{x\to 0} x^2 \sin\frac{1}{x} = 0$$

Q Why can't you use the Squeeze Theorem as in exercise 29 to prove that $\lim_{x\to 0} x^2 \sec\left(\frac{1}{x}\right) = 0$? Explore this limit graphically.

Answer:

- The function $\sec x$ has vertical asymptotes (discontinuities) at $x = \frac{\pi}{2} + k\pi$ (i.e., at the odd multiples of $\frac{\pi}{2}$), so $\sec x$ (and hence $\sec\left(\frac{1}{x}\right)$) is **not bounded.**
- Therefore, we cannot find a finite upper bound $h(x)$ and a finite lower bound $g(x)$ that contain $x^2 \sec\left(\frac{1}{x}\right)$ near $x = 0$. Hence, the Squeeze Theorem **cannot be used.**

Graphical Exploration



Q 31 Use the Squeeze Theorem to prove that $\lim_{x\to 0^+} \sqrt{x}\cos^2(1/x) = 0$.
Identify the functions f and h, show graphically that $f(x) \le \sqrt{x}\cos^2(1/x) \le h(x)$ for all $x > 0$, and justify $\lim_{x\to 0^+} f(x) = 0$ and $\lim_{x\to 0^+} h(x) = 0$.

1 Find a bounding inequality

Since $0 \le \cos^2\left(\frac{1}{x}\right) \le 1$, and $\sqrt{x} > 0$ for $x > 0$,

multiply by $\sqrt{x}$: $0 \le \sqrt{x}\cos^2\left(\frac{1}{x}\right) \le \sqrt{x}$

2 Identify the functions

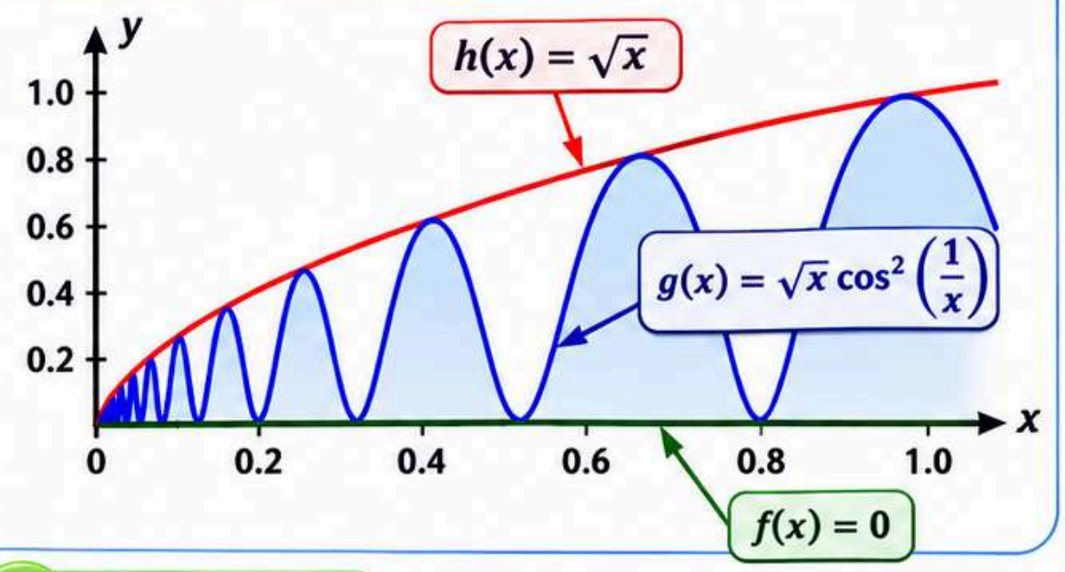
- $f(x) = 0$
- $g(x) = \sqrt{x}\cos^2\left(\frac{1}{x}\right)$
- $h(x) = \sqrt{x}$

$f(x) \le g(x) \le h(x)$ for all $x > 0$

3 Find the limits of the bounding functions

- $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} 0 = 0$
- $\lim_{x\to 0^+} h(x) = \lim_{x\to 0^+} \sqrt{x} = 0$

4 Graphical illustration



Conclusion

By the Squeeze Theorem, since $f(x) \le g(x) \le h(x)$ for $x > 0$ and $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} h(x) = 0$, we conclude that

$$\lim_{x\to 0^+} \sqrt{x}\cos^2\left(\frac{1}{x}\right) = 0$$

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Q3	Use the squeeze theorem to find limits	Squeeze Theorem	82

Q If $|g(x) - 4| \le 2(2 - x)$ is true for all values of x, evaluate $\lim\limits_{x \to 2} g(x)$.

Solution:

1 **Rewrite the inequality:**

$$|g(x) - 4| \le 2(2 - x)$$
$$\Leftrightarrow \quad -2(2 - x) \le g(x) - 4 \le 2(2 - x)$$

2 **Add 4 to all parts:**

$$4 - 2(2 - x) \le g(x) \le 4 + 2(2 - x)$$
$$2x \le g(x) \le 8 - 2x$$

3 **Find the limits of the bounding functions:**

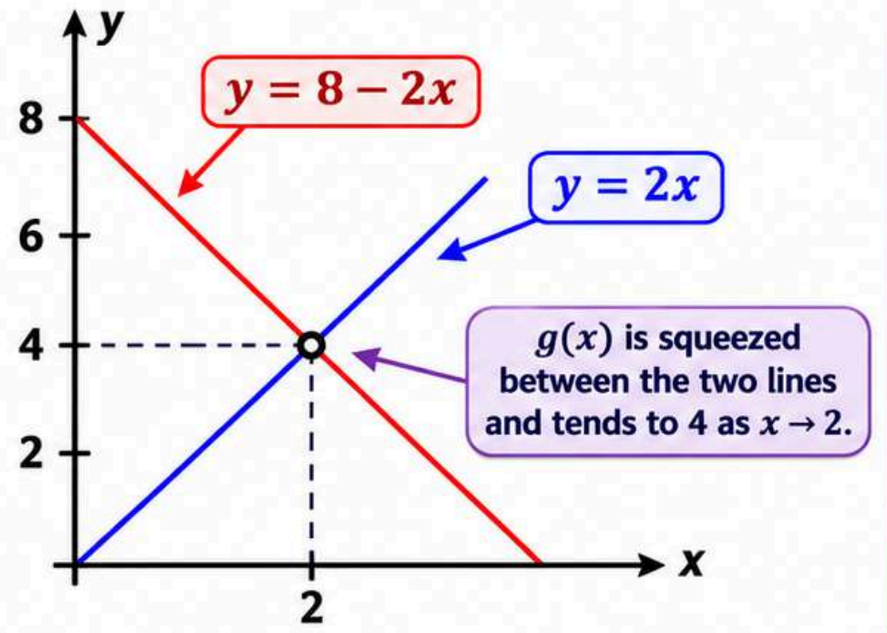
- $\lim\limits_{x \to 2} 2x = 2(2) = 4$
- $\lim\limits_{x \to 2} (8 - 2x) = 8 - 2(2) = 4$

4 **Apply the Squeeze Theorem:**

Since $2x \le g(x) \le 8 - 2x$ for all x, and $\lim\limits_{x \to 2} 2x = \lim\limits_{x \to 2} (8 - 2x) = 4$, we conclude that

$$\lim_{x \to 2} g(x) = 4$$

Graphical Illustration



Answer: $\lim\limits_{x \to 2} g(x) = 4$

Q4	Use the Intermediate Value Theorem to determine the zeros of a function	استخدام نظرية القيمة الوسطية لتحديد أصفار الدالة	Intermediate Value Theorem	93	نظرية القيمة المتوسطة	95

Intermediate Value Theorem (IVT)

If f is continuous on the closed interval $[a, b]$ and $f(a)$ and $f(b)$ have **opposite signs,** that is, $f(a) \cdot f(b) < 0,$

then there exists **at least one** number c in (a, b) such that $f(c) = 0.$

➡ The graph of f must cross the x-axis ***at least once*** between a and b.

Verification Steps

1. ***Check continuity:*** Verify that f is continuous on $[a, b]$.
2. ***Calculate endpoint values:*** Find $f(a)$ and $f(b)$.
3. ***Confirm opposite signs:*** Check that $f(a) \cdot f(b) < 0$ (i.e., $f(a)$ and $f(b)$ have different signs).
4. ***Conclude:*** There is at least one root c in (a, b) such that $f(c) = 0$.

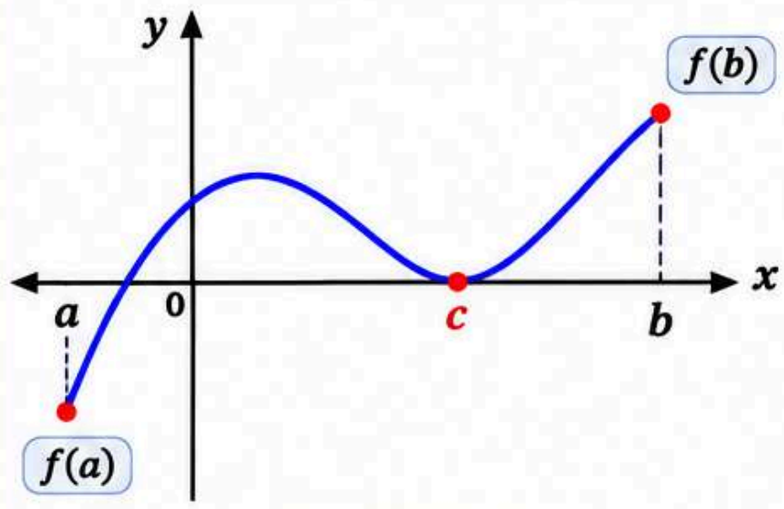
Important Notes

- ***Continuity*** **is essential:** The theorem applies only if f is continuous on the closed interval $[a, b]$.
- ***Opposite endpoint signs*** guarantee ***at least one root***, not necessarily exactly one (the graph may cross the x-axis more than once).
- If $f(a)$ and $f(b)$ have the ***same sign*** $(f(a) \cdot f(b) > 0)$, this theorem alone cannot guarantee a root in (a, b).

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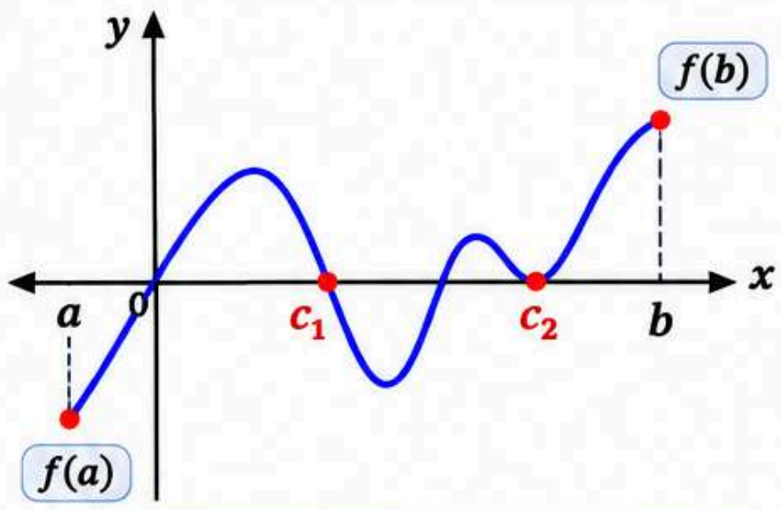
Graphical Illustration

One crossing (at least one root)



The graph crosses the x-axis **once** at $x = c$ $(f(c) = 0)$.

More than one crossing (still at least one root)



The graph crosses the x-axis **more than once** (e.g., at $x = c_1$ and $x = c_2$).

Example 33

Question

Use the Intermediate Value Theorem to verify that $f(x)$ has a zero in the given interval. Then use the method of bisections to find an interval of length 1/32 that contains the zero.

$$f(x) = x^2 - 7, \qquad [2, 3].$$

Solution

1 Verify (IVT):

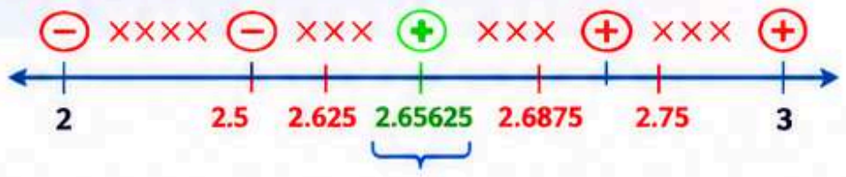
- $f(2) = 2^2 - 7 = -3 < 0.$
- $f(3) = 3^2 - 7 = 2 > 0.$
- f is continuous and changes sign on $[2, 3]$, so by the IVT, f has a zero in $[2, 3]$.

2 Bisection Method:

Iteration	Midpoint m	$f(m)$	Sign	New interval
1	2.5	−0.75	−	[2.5, 3]
2	2.75	0.5625	+	[2.5, 2.75]
3	2.625	−0.109375	−	[2.625, 2.75]
4	2.6875	0.2265625	+	[2.625, 2.6875]
5	2.65625	0.05615234375	+	[2.625, 2.65625]

3 Result:

- The sign change is between 2.625 and 2.55625.
- The interval length is $2.65625 - 2.625 = 0.03125 = \frac{1}{32}$.



Final interval: $[2.625, 2.65625]$.

Example 36

Question

Use the Intermediate Value Theorem to verify that $f(x)$ has a zero in the given interval. Then use the method of bisections to find an interval of length 1/32 that contains the zero.

$$f(x) = e^x + x, \qquad [-1, 0].$$

Solution

1 Verify (IVT):

- $f(-1) = e^{-1} - 1 \approx 0.3679 - 1 = -0.6321 < 0.$
- $f(0) = e^0 + 0 = 1 > 0.$
- f is continuous and changes sign on $[-1, 0]$, so by the IVT, f has a zero in $[-1, 0]$.

2 Bisection Method:

Iteration	Midpoint m	$f(m)$	Sign	New interval
1	−0.5	0.1065	+	[−1, −0.5]
2	−0.75	−0.2776	−	[−0.75, −0.5]
3	−0.625	−0.0897	−	[−0.625, −0.5]
4	−0.5625	0.0087	+	[−0.625, −0.5625]
5	−0.59375	−0.0399	−	[−0.59375, −0.5625]

3 Result:

- The sign change is between −0.59375 and −0.5625.
- The interval length is $-0.5625 - (-0.59375) = 0.03125 = \frac{1}{32}$.



Final interval: $[-0.59375, -0.5625]$.





بعد التحديث

Q4	Find limits for polynomials, rational, and trigonometric functions using theorems	إيجاد النهايات لكثيرات الحدود والدوال الكسرية والدوال المثلثية باستخدام النظريات	(7-14)	85	(7-16)	87

7 $\lim_{x\to 2} \dfrac{x^2-x-2}{x^2-4}$

Solution: $= \dfrac{2^2-2-2}{2^2-4} = \dfrac{0}{0} = 0$

(A) $\frac{3}{4}$ ✓ (B) does not exist (C) -3 (D) 14

8 $\lim_{x\to 1} \dfrac{x^3-1}{x^2+2x-3}$

Solution: $= \dfrac{1^3-1}{1^2+2(1)-3} = \dfrac{0}{0} = 0$

(A) $\frac{3}{4}$ ✓ (B) does not exist (C) -3 (D) 14

9 $\lim_{x\to 0} \dfrac{\sin x}{\tan x}$

Solution: $\lim_{x\to 0} \sin x \cdot \dfrac{1}{\tan x} = \lim_{x\to 0} \dfrac{\sin x}{\frac{\sin x}{\cos x}} = \lim_{x\to 0} \cos x = \cos(0) = 1$

(A) 1 ✓ (B) does not exist (C) -3 (D) 14

10 $\lim_{x\to 0} \dfrac{\tan x}{x}$

Solution: $\lim_{x\to 0} \dfrac{\tan x}{x} = \lim_{x\to 0} \dfrac{\sin x}{x} \cdot \dfrac{1}{\cos x} = 1 \cdot 1 = 1$

(A) 1 ✓ (B) does not exist (C) -3 (D) -3

11 $\lim_{x\to 0} \dfrac{xe^{-2x+1}}{x^2+x}$

Solution: $= \lim_{x\to 0} \dfrac{e^{-2x+1}}{x+1} = \dfrac{e^1}{1} = e$

(A) e ✓ (B) does not exist (C) -3 (D) 14

12 QUESTION 12

$\lim_{x\to 0} x^2 \csc^2 x$

SOLUTION

Recall that csc x = 1/sin x.
$x^2 \csc^2 x = x^2 / \sin^2 x = (x / \sin x)^2$
Using $\lim_{x\to 0} (\sin x / x) = 1$, we get
$(x / \sin x)^2 \to 1^2 = 1$.

A 1 B Does not exist C −3 D −1

13. $\lim_{x\to 0} \dfrac{\sqrt{x+4}-2}{x}$

SOLUTION

Direct substitution gives 0/0, so multiply by the conjugate.
Multiply numerator and denominator by $\sqrt{(x + 4)} + 2$.
The numerator becomes $(x + 4) - 4 = x$.
Cancel x: $1 / (\sqrt{(x + 4)} + 2)$.
Substitute x = 0: $1 / (2 + 2) = 1/4$.

A 1/4 B Does not exist C −3 D −1

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14. $\lim_{x \to 0} \dfrac{2x}{3 - \sqrt{x+9}}$

SOLUTION

Direct substitution gives 0/0. Multiply by the conjugate.
Multiply by $(3 + \sqrt{(x + 9)}) / (3 + \sqrt{(x + 9)})$.
Denominator: $9 - (x + 9) = -x$.
Cancel x: $-2(3 + \sqrt{(x + 9)})$.
Substitute $x = 0$: $-2(3 + 3) = -12$.

A −12 | B Does not exist | C −3 | D −6

15. $\lim_{x \to 1} \dfrac{x-1}{\sqrt{x}-1}$

16. $\lim_{x \to 4} \dfrac{x^3 - 64}{x - 4}$

Q5	Study the continuity of a function at a given point	دراسة اتصال الدالة عند نقطة	(21-28)	95	(21-28)	97

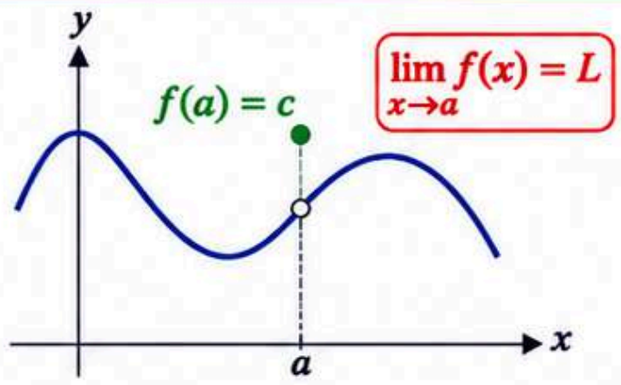
Types of Discontinuity

Definition: A function is **discontinuous** at $x = a$ if it is not continuous at that point, i.e., its limit does not agree with its function value, or the limit does not exist.

1 Removable Discontinuity

- The limit exists (finite), but the function value is either not defined or different from the limit.
- The graph has a "hole" at $x = a$.

Condition: $\lim_{x \to a} f(x) = L$ but $f(a)$ is not defined or $f(a) \neq L$.

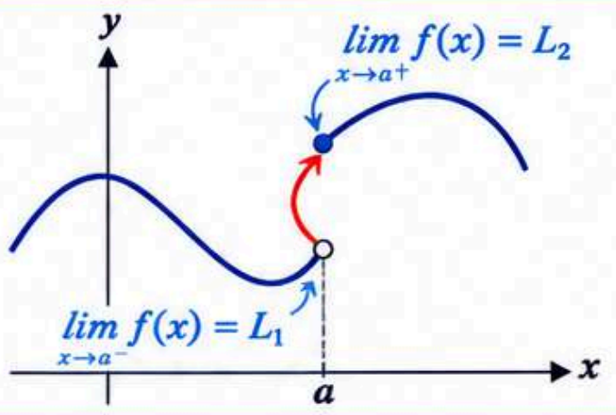


Key Idea:
The two one-sided limits are equal, so the limit exists, but the function is not continuous due to a hole or a wrong value at $x = a$.

2 Jump Discontinuity

- The left-hand and right-hand limits both exist, but they are different.
- The graph "jumps" from one value to another at $x = a$.

Condition: $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$.

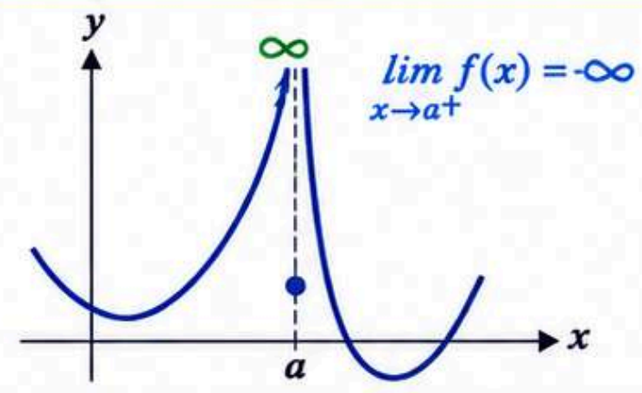


Key Idea:
The function approaches two different values from the left and the right, so the two-sided limit does not exist.

3 Infinite Discontinuity

- The function grows without bound (to $+\infty$ or $-\infty$) as x approaches a.
- The graph has a vertical asymptote at $x = a$.

Condition: $\lim_{x \to a} f(x) = +\infty$ or $\lim_{x \to a} f(x) = -\infty$.



Key Idea:
The function becomes arbitrarily large (infinite) near $x = a$, so the limit does not exist.

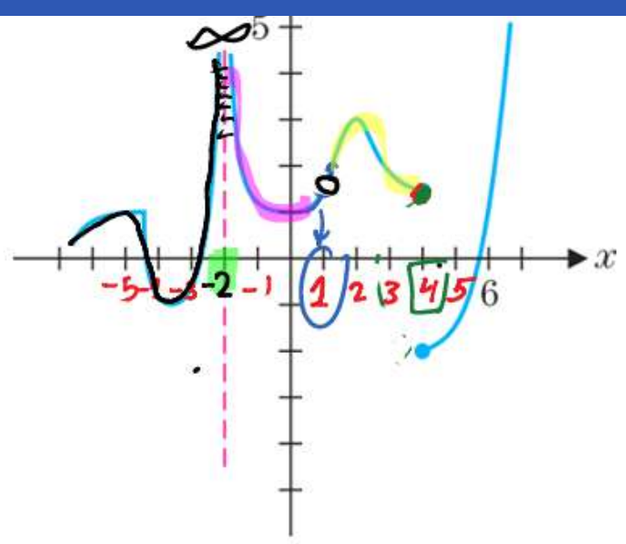
Summary

Type of Discontinuity	Behavior of the Limit at $x = a$	Graph Feature
Removable	Limit exists (finite), but $f(a)$ is not defined or $f(a) \neq L$.	A hole (can be filled).
Jump	Left-hand limit ≠ right-hand limit (both finite).	A jump between two values.
Infinite	Limit is $+\infty$ or $-\infty$.	A vertical asymptote.

Important Notes

- A function is continuous at $x = a$ if: $\lim_{x \to a} f(x)$ exists and equals $f(a)$.
- If the limit exists but $f(a)$ is missing or different → removable **discontinuity**.
- If the one-sided limits are different → jump **discontinuity**.
- If the function goes to $\pm\infty$ → infinite discontinuity.

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Solution

Point	Type of Discontinuity	Reason
$x = -2$	**Infinite Discontinuity**	The function grows without bound (vertical asymptote) as $x \to -2$.
$x = 1$	**Removable Discontinuity**	The limit exists, but the function is not defined at $x = 1$ (a hole).
$x = 4$	**Jump Discontinuity**	The left-hand and right-hand limits are different (the graph jumps at $x = 4$).

Answer

- **Discontinuity points:** $x = -2$, $x = 1$, $x = 4$.
- **Their types:**
  - $x = -2$: Infinite discontinuity
  - $x = 1$: Removable discontinuity
  - $x = 4$: Jump discontinuity

Intervals of Continuity

- The function is continuous on each part of its domain between the points of discontinuity.

Intervals: $(-\infty, -2)$, $(-2, 1)$, $(1, 4)$, $(4, +\infty)$

Connected (Continuous) Functions

Functions that are Continuous Everywhere

Definition: A function is continuous everywhere on $\mathbb{R}$ if it is continuous at every real number.

Examples:

1. **Polynomials:** $x^4 + x^3 + x^2 - \dots$ — Continuous on $\mathbb{R}$.
2. **Trigonometric functions:** $\cos x$, $\sin x$, $\tan x$ — Continuous on $\mathbb{R}$.
3. **Exponential function:** e^x — Continuous on $\mathbb{R}$.
5. **Root functions (n odd):** $\sqrt[n]{x}$ (n is odd) — Continuous on $\mathbb{R}$.

4

Note: The domain of each of the above functions is $\mathbb{R} = (-\infty, \infty)$.

Functions that are Continuous on a Specific Interval

Definition: A function is continuous on an interval I if it is continuous at every point in I.

Examples:

1	$\sqrt[n]{x}$ (n is even)	Domain and interval of continuity: $[0, \infty)$, i.e., $x \geq 0$.
2	$\cos^{-1} x$, $\sin^{-1} x$	Continuous on $-1 \leq x \leq 1$.
3	$\ln x$	Continuous on $(0, \infty)$, i.e., $x > 0$.

Domains and Intervals of Continuity (Summary)

Function	Domain (and Interval of Continuity)
Polynomial (e.g., $x^4 + x^3 + x^2$)	$\mathbb{R} = (-\infty, \infty)$
$\sin x$, $\cos x$, $\tan x$	$\mathbb{R} = (-\infty, \infty)$
e^x	$\mathbb{R} = (-\infty, \infty)$
$\sqrt[n]{x}$ (n even)	$[0, \infty)$
$\sqrt[n]{x}$ (n odd)	$\mathbb{R} = (-\infty, \infty)$
$\cos^{-1} x$, $\sin^{-1} x$	$[-1, 1]$
$\ln x$	$(0, \infty)$

Discontinuity Points of Rational Functions

A rational function is a ratio of two polynomials, $f(x) = \frac{P(x)}{Q(x)}$. Discontinuity points occur where $Q(x) = 0$.

Types of Discontinuity

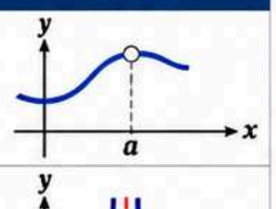
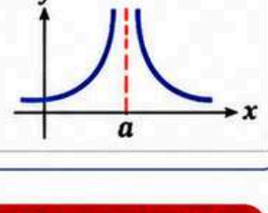
1 Removable Discontinuity (Hole)

- Occurs when a common factor $(x - a)$ cancels from the numerator and denominator.
- The limit exists (finite), but the function is not defined at $x = a$.
- The graph has a hole at the point (a, L).

2 Non-Removable Discontinuity (Vertical Asymptote)

- Occurs when $Q(x) = 0$ and there is no common factor to cancel.
- The function grows without bound as $x \to a$.
- The graph has a vertical asymptote $x = a$.

Properties of the Limit at a Discontinuity

Type	$\lim_{x \to a} f(x)$	Graph
Removable (Hole)	Exists (finite)	
Non-Removable (Vertical Asymptote)	$\pm\infty$ (does not exist as a finite value)	

Example 1: Removable Discontinuity (Hole)

$$\lim_{x \to 1} \frac{x^2 + 2x - 3}{x - 1}$$

Solution

- Factor the numerator: $x^2 + 2x - 3 = (x - 1)(x + 3)$
- Cancel the common factor $(x - 1)$: $\frac{x^2 + 2x - 3}{x - 1} = \frac{(x - 1)(x + 3)}{x - 1} = x + 3 \quad (x \neq 1)$
- Evaluate the limit: $\lim_{x \to 1} (x + 3) = 1 + 3 = 4$

- The function is not defined at $x = 1$, but the limit exists and equals 4.
- So, $x = 1$ is a removable discontinuity (a hole) at $(1, 4)$.

How to Determine the Type of Discontinuity

1. Find the values that make the denominator $Q(x) = 0$.
2. Factor the numerator and denominator.
3. Check for a common factor $(x - a)$:
   - If it cancels → removable discontinuity (hole).
   - If it does not cancel → non-removable (vertical asymptote).

Example 2: Non-Removable Discontinuity

$$f(x) = \frac{x + 2}{x - 1}$$

Solution

- The denominator is zero at $x = 1$.
- There is no common factor to cancel.
- So, $x = 1$ is a non-removable discontinuity (vertical asymptote $x = 1$)).

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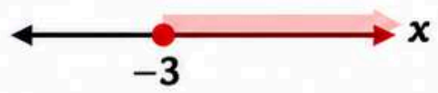


Determine the Intervals on Which f is Continuous

21 $f(x) = \sqrt{x+3}$

Solution

1. The function is a square root function, so its radicand must be nonnegative.
2. $x + 3 \geq 0 \Rightarrow x \geq -3.$



Interval of continuity: $[-3, \infty)$

Answer: A

22 $f(x) = \sqrt{x^2 - 4}$

Solution

1. Require the radicand to be nonnegative.
2. $x^2 - 4 \geq 0$
3. $(x-2)(x+2) \geq 0$
   $\Rightarrow x \leq -2$ or $x \geq 2.$



Interval of continuity: $(-\infty, -2] \cup [2, \infty)$

Answer: A

23 $f(x) = \sqrt[3]{x+2}$

Solution

1. The cube root function $\sqrt[3]{x}$ is defined for all real numbers.
2. Adding a constant $(+2)$ does not restrict the domain.
3. So the domain is $\mathbb{R}$, and the function is continuous on $\mathbb{R}$.

Interval of continuity: $(-\infty, \infty)$

Answer: A

24 $f(x) = (x-1)^{3/2}$

Solution

1. Rewrite the exponent:
   $(x-1)^{3/2} = \sqrt{(x-1)^3}.$
2. The radicand must be nonnegative:
   $(x-1)^3 \geq 0 \Rightarrow x - 1 \geq 0$
   $\Rightarrow x \geq 1.$



Interval of continuity: $[1, \infty)$

Answer: A

Determine the Intervals on Which f is Continuous

25 $f(x) = \sin^{-1}(x+2)$

A $[-3, -1]$ B $(-2, 0)$ C $(-3, -1)$ D $(-\infty, \infty)$

Solution

1. For $\sin^{-1}u$, the domain is $-1 \leq u \leq 1.$
2. Let $u = x + 2$:
   $-1 \leq x + 2 \leq 1.$
3. Subtract 2:
   $-3 \leq x \leq -1.$

Interval of continuity: $[-3, -1]$



Answer: A

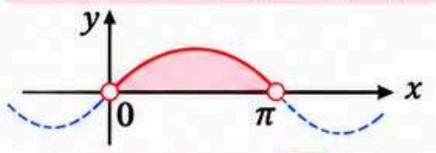
26 $f(x) = \ln(\sin x)$

A $\sin x > 0$ B $\sin x \geq 0$ C $\sin x \leq 0$ D $\sin x \neq 1$

Solution

1. For $\ln u$, we need $u > 0.$
2. Let $u = \sin x$:
   $\sin x > 0.$
3. This occurs when x is in the first or second quadrant:
   $x \in (0, \pi).$

Interval of continuity:
$\sin x > 0$ *(i.e.,* $x \in (0, \pi)$*)*



Answer: A

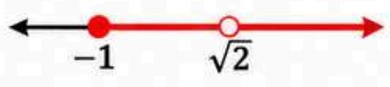
27 $f(x) = \dfrac{\sqrt{x+1} + e^x}{x^2 - 2}$

A $x \neq \sqrt{2}, [-1, \infty)$ B $x \neq 0, [\sqrt{2}, \infty)$ C $x \neq \sqrt{2}, [0, \infty)$ D $x \neq \sqrt{3}, [2, \infty)$

Solution

1. Domain of $\sqrt{x+1}$: $x + 1 \geq 0 \Rightarrow x \geq -1.$
2. e^x is defined for all real numbers.
3. Denominator: $x^2 - 2 \neq 0 \Rightarrow x \neq \pm\sqrt{2}.$
4. Combine the conditions:
   $x \geq -1$ and $x \neq \sqrt{2}.$

Interval of continuity: $x \in [-1, \infty) \setminus \{\sqrt{2}\}$



Answer: A

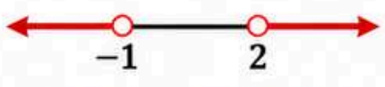
28 $f(x) = \dfrac{\ln(x^2 - 1)}{\sqrt{x^2 - 2x}}$

A $(-\infty, -1), (2, \infty)$ B $(-\infty, 0), (1, \infty)$ C $(-\infty, -4), (2, \infty)$ D $(-\infty, 1), (2, \infty)$

Solution

1. For $\ln(x^2 - 1)$, we need: $x^2 - 1 > 0$
   $\Rightarrow (x-1)(x+1) > 0 \Rightarrow x < -1$ or $x > 1.$
2. For $\sqrt{x^2 - 2x}$, we need: $x^2 - 2x > 0$
   $\Rightarrow x(x-2) > 0 \Rightarrow x < 0$ or $x > 2.$
3. Intersect the two conditions:
   $(x < -1$ or $x > 1) \cap (x < 0$ or $x > 2)$
   $= (-\infty, -1) \cup (2, \infty).$

Interval of continuity: $(-\infty, -1) \cup (2, \infty)$



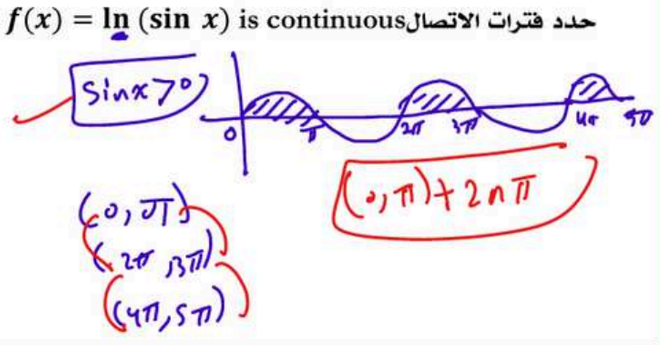
Answer: A

Key Takeaways:

- $\sqrt{g(x)}$ is continuous where $g(x) \geq 0.$
- $\sqrt[3]{g(x)}$ is continuous for all real $x.$
- $\ln g(x)$ is continuous where $g(x) > 0.$
- $\sin^{-1} g(x)$, $\cos^{-1} g(x)$ are continuous where $-1 \leq g(x) \leq 1.$
- A rational function is continuous on its domain (denominator $\neq 0$).
- The function is continuous on the intersection of the conditions from all parts.

26. Determine the intervals on which $f(x) = \ln(\sin x)$ is continuous حدد فترات الاتصال

a) $(2n\pi, (2n+1)\pi)$ for integer n
b) $(n\pi, (n+1)\pi)$ for integer n
c) All real numbers
d) Nowhere continuous



39. Determine values of a and b that make the given function continuous.

أوجد قيمة a و b التي تجعل الدالة متصلة

$$f(x) = \begin{cases} \frac{2\sin x}{x} & if\ x < 0 \\ a & if\ x = 0 \\ b\cos x & if\ x > 0 \end{cases}$$

$\lim_{x \to 0^-} f(x) = \lim_{x \to 0} \frac{2\sin x}{x} = 2(1) = 2$

$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} b\cos x = b(1) = b$

$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x)$

$b = 2$

$f(0) = \lim_{x \to 0} f(x)$ $f(x)$ is Cont

$a = 2$

40. Determine values of a and b that make the given function continuous.

أوجد قيمة a و b التي تجعل الدالة متصلة

$$f(x) = \begin{cases} ae^x + 1 & if\ x < 0 \\ \sin^{-1}(\frac{x}{2}) & if\ 0 \leq x \leq 2 \\ x^2 - x + b & if\ x > 2 \end{cases}$$

at $x = 0$

$\lim_{x \to 0^-} ae^x + 1 = ae^0 + 1 = a + 1$

$\lim_{x \to 0^+} \sin^{-1}(\frac{x}{2}) = \sin^{-1}(\frac{0}{2}) = 0$

$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x)$ $f(x)$ Cont

$a + 1 = 0$

$a = -1$

at $x = 2$

$\lim_{x \to 2^-} \sin^{-1}(\frac{x}{2}) = \sin^{-1}(\frac{2}{2}) = \frac{\pi}{2}$

$\lim_{x \to 2^+} x^2 - x + b = (2)^2 - 2 + b = 2 + b$

$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x)$ $f(x)$ is Cont

$2 + b = \frac{\pi}{2}$

$b = \frac{\pi}{2} - 2$

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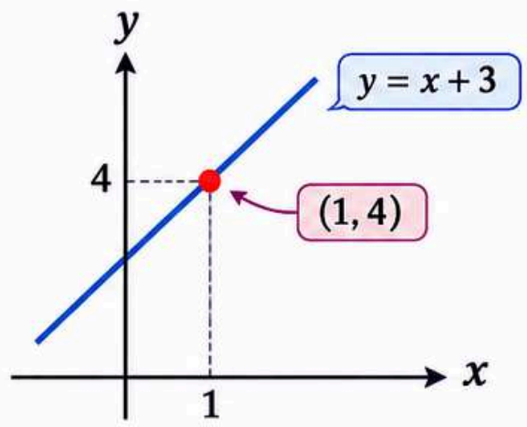
Q6	Use the continuity properties to study the continuity of a function or a composition of functions at a given point	استخدم خصائص الاتصال لدراسة اتصال الدالة أو مجموعة الدوال عند نقطة معينة	Example 4.2	89	Example 4.2	91

Example 4.2 Removing a Hole in the Graph

Extend the function from Example 4.1 to make it continuous everywhere by defining it at a single point.

Example

Determine where $f(x) = \dfrac{x^2 + 2x - 3}{x - 1}$ is continuous.



Solution

1. **Factor the numerator:**
$x^2 + 2x - 3 = (x + 3)(x - 1)$.

2. **Simplify the function (for $x \neq 1$):**
$f(x) = \dfrac{(x+3)(x-1)}{x-1} = x + 3, \quad x \neq 1.$

3. **Define f at $x = 1$ using the limit:**
$\lim\limits_{x \to 1} f(x) = \lim\limits_{x \to 1} (x + 3) = 1 + 3 = 4.$

4. **Extend the function:**
$$g(x) = \begin{cases} \dfrac{x^2 + 2x - 3}{x - 1}, & x \neq 1, \\ 4, & x = 1. \end{cases}$$

5. **Conclusion:** g is continuous for all real numbers.

Discussion

- The original function $f(x) = \dfrac{x^2 + 2x - 3}{x - 1}$ is not defined at $x = 1$, so it has a **hole** at $(1, 4)$.
- By defining $g(1) = 4$, we fill in the hole, and the new function $g(x)$ becomes continuous everywhere.

Key Takeaways

- A **hole (removable discontinuity)** occurs when a factor **cancels** from the numerator and denominator.
- To remove the hole, define the function at that point using the limit value.
- The result is a **continuous** function on all real numbers.

في التمارين 14–1، حدّد أين تكون f متصلة. إذا كان ممكنًا، توسّع في f كما في المثال 4.2 إلى دالة جديدة متصلة على نطاق أكبر.

4. $f(x) = \dfrac{4x}{x^2 + x - 2}$

$x^2 + x - 2 = 0$: أصفار المقام

$(x-1)(x+2)$ نقاط انفصال

$x = 1 \quad x = -2$

$(-\infty, \infty)$ ما عدا $x = 1, x = -2$

$f(x) = \dfrac{4x}{x^2 + x - 2}$

$= \dfrac{4x}{(x-1)(x+2)}$ لا يوجد إزالة لنقاط انفصال

$$f(x) = \begin{cases} \dfrac{4x}{x^2 + x - 2}, & x \neq 1, \; x \neq -2 \\ \text{متصلة} \end{cases}$$

5. $f(x) = \dfrac{4x}{x^2 + 4}$

$x^2 + 4 = 0$: أصفار المقام

$x^2 = -4$

$x = \pm\sqrt{-4}$

الدالة متصلة على $\mathbb{R}$

27. $f(x) = \dfrac{\sqrt{x+1} + e^x}{x^2 - 2}$

$[-1, \infty)$ مجال البسط $\sqrt{x+1}$ ، $x + 1 \geq 0$، $x \geq -1$

مجال $\mathbb{R}$ للمقام

أصفار المقام (نقاط انفصال)

$[-1, \infty) / +\sqrt{2}$

$(-\infty, -\sqrt{2}) \cup (-1, \sqrt{2}) \cup (\sqrt{2}, \infty)$

$x^2 - 2 = 0$: أصفار المقام

$x^2 = 2$

$x = \pm\sqrt{2}$

$x = \sqrt{2}, \; x = -\sqrt{2}$

في التمارين 14–1، حدّد أين تكون f متصلة. إذا كان ممكنًا، توسّع في f كما في المثال 4.2 إلى دالة جديدة متصلة على نطاق أكبر.

3. $f(x) = \dfrac{x-1}{x^2 - 1}$ مجال $\mathbb{R}$، مجال $\mathbb{R}$

$x^2 - 1 = 0$: أصفار المقام

$x^2 = 1$

$x = \pm 1$

$(-\infty, \infty)$ ما عدا $x = \pm 1$

$(x-1) \quad (x+1)$

$x = 1 \quad x = -1$

نقاط انفصال

$$f(x) = \begin{cases} \dfrac{x-1}{x^2-1}, & x \neq \pm 1 \\ \dfrac{1}{2}, & x = 1 \end{cases}$$

$f(x) = \dfrac{x-1}{x^2-1} = \dfrac{x-1}{(x-1)(x+1)} \Rightarrow f(x) = \dfrac{1}{x+1}$

$x = 1 \quad x = -1$

$f(1) = \dfrac{1}{1+1} = \dfrac{1}{2}$

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@Badr_academy25
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Q7	Find infinite limits and limits at infinity	إيجاد النهايات التي تؤول إلى اللانهاية والنهايات عند اللانهاية	(9-14)	106	(9-14)	108

10 $\lim_{x\to\infty} \dfrac{2x^2 - x + 1}{4x^2 - 3x - 1}$

(A) $\frac{1}{2}$ (B) $-\infty$ (C) does not exist (D) ∞

Solution

1. **Divide by the highest power x^2:**
$$\lim_{x\to\infty} \frac{2x^2 - x + 1}{4x^2 - 3x - 1} = \lim_{x\to\infty} \frac{2 - \frac{1}{x} + \frac{1}{x^2}}{4 - \frac{3}{x} - \frac{1}{x^2}}.$$
2. **Evaluate the limit:**
As $x \to \infty$, $\frac{1}{x} \to 0$ and $\frac{1}{x^2} \to 0$, so $\lim_{x\to\infty} \frac{2 - \frac{1}{x} + \frac{1}{x^2}}{4 - \frac{3}{x} - \frac{1}{x^2}} = \frac{2}{4} = \frac{1}{2}$.

Answer: (A) $\frac{1}{2}$

11 $\lim_{x\to-\infty} \dfrac{-x}{\sqrt{4 + x^2}}$

(A) 1 (B) $-\infty$ (C) does not exist (D) ∞

Solution

1. **Rewrite the denominator:** $\sqrt{4 + x^2} = \sqrt{x^2\left(1 + \frac{4}{x^2}\right)} = |x|\sqrt{1 + \frac{4}{x^2}}$.
2. **Consider $x \to -\infty$ (so $x < 0$ and $|x| = -x$):**
$$\frac{-x}{\sqrt{4 + x^2}} = \frac{-x}{|x|\sqrt{1 + \frac{4}{x^2}}} = \frac{-x}{(-x)\sqrt{1 + \frac{4}{x^2}}} = \frac{1}{\sqrt{1 + \frac{4}{x^2}}}.$$
3. **Evaluate the limit:**
As $x \to -\infty$, $\frac{4}{x^2} \to 0$, so $\lim_{x\to-\infty} \frac{-x}{\sqrt{4 + x^2}} = \frac{1}{\sqrt{1 + 0}} = 1$.

Answer: (A) 1

Key Takeaways:

- For rational functions $\frac{P(x)}{Q(x)}$ as $x \to \pm\infty$:
  - If $\deg P < \deg Q \Rightarrow 0$.
  - If $\deg P = \deg Q \Rightarrow$ ratio of leading coefficients.
  - If $\deg P > \deg Q \Rightarrow \pm\infty$.
- $\sqrt{x^2} = |x|$ and $\sqrt{x^2 + a} = |x|\sqrt{1 + \frac{a}{x^2}}$.
- As $x \to \infty$ or $x \to -\infty$, $\frac{1}{x} \to 0$ and $\frac{1}{x^2} \to 0$.
- Simplify the expression by dividing by the highest power of x in the numerator and denominator.

12 $\lim_{x\to-\infty} \dfrac{2x^2 - 1}{4x^3 - 5x - 1}$

(A) 0 (B) $-\infty$ (C) does not exist (D) ∞

Solution

1. The highest powers are x^2 (numerator) and x^3 (denominator).
2. **Divide by x^3:** $\lim_{x\to-\infty} \frac{2x^2 - 1}{4x^3 - 5x - 1} = \lim_{x\to-\infty} \frac{\frac{2}{x} - \frac{1}{x^3}}{4 - \frac{5}{x^2} - \frac{1}{x^3}}$.
3. As $x \to -\infty$, $\frac{2}{x}, \frac{1}{x^2}, \frac{1}{x^3} \to 0$, so the limit is $\frac{0}{4} = 0$.

Answer: (A) 0

13 $\lim_{x\to\infty} \ln\left(\dfrac{x^2 + 1}{x - 3}\right)$

(A) ∞ (B) $-\infty$ (C) does not exist (D) 3

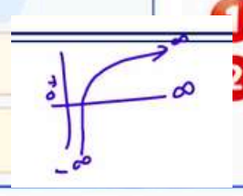
Solution

1. **Simplify the expression inside the logarithm:** $\frac{x^2 + 1}{x - 3} = \frac{x^2\left(1 + \frac{1}{x^2}\right)}{x\left(1 - \frac{3}{x}\right)} = x\frac{1 + \frac{1}{x^2}}{1 - \frac{3}{x}}$.
2. As $x \to \infty$, $\frac{1}{x^2} \to 0$ and $\frac{3}{x} \to 0$, so $\frac{x^2 + 1}{x - 3} \sim x \to \infty$.
3. Therefore, $\lim_{x\to\infty} \ln\left(\frac{x^2 + 1}{x - 3}\right) = \ln(\infty) = \infty$.

Answer: (A) ∞

14 $\lim_{x\to0^+} \ln(x \sin x)$

(A) $-\infty$ (B) ∞ (C) does not exist (D) 3



Solution

1. As $x \to 0^+$, $\sin x \sim x$ (or $\sin x \to 0^+$), so $x \sin x \sim x^2 \to 0^+$.
2. The logarithm of a positive number that approaches 0 is $-\infty$, so $\lim_{x\to0^+} \ln(x \sin x) = -\infty$.

Answer: (A) $-\infty$

Key Takeaways:

- $\lim_{x\to\pm\infty} \frac{P(x)}{Q(x)} = \begin{cases} 0, & \deg P < \deg Q \\ c, & \deg P = \deg Q \\ \pm\infty, & \deg P > \deg Q \end{cases}$
- $\ln x \to \infty$ as $x \to \infty$.
- $\ln x \to -\infty$ as $x \to 0^+$.
- $\sin x \sim x$ as $x \to 0$.
- For $\ln(g(x))$, first find the limit of $g(x)$, then use the behavior of $\ln x$.

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اسئلة اضافية

Find the limit: $\lim_{x\to 0^+} e^{-2/x^3}$

A) 0 B) 1 C) ∞ D) $-\infty$

Solution:

- As $x \to 0^+$, we have $x^3 \to 0^+$, so $\frac{1}{x^3} \to +\infty$.
- Thus, $-\frac{2}{x^3} \to -\infty$.
- Therefore,

$$\lim_{x\to 0^+} e^{-2/x^3} = e^{-\infty} = 0.$$

Answer: A

Find the limit: $\lim_{x\to \infty} e^{-(x+1)/(x^2+2)}$

A) 0 B) 1 C) ∞ D) $-\infty$

Solution:

- Divide numerator and denominator by x^2:

$$\frac{x+1}{x^2+2} = \frac{\frac{x}{x^2}+\frac{1}{x^2}}{1+\frac{2}{x^2}} = \frac{\frac{1}{x}+\frac{1}{x^2}}{1+\frac{2}{x^2}}.$$

- As $x \to \infty$, $\frac{1}{x} \to 0$ and $\frac{1}{x^2} \to 0$, so

$$\frac{x+1}{x^2+2} \to \frac{0+0}{1+0} = 0.$$

- Therefore, $\lim_{x\to\infty} e^{-(x+1)/(x^2+2)} = e^0 = 1.$

Answer: B

Find the limit: $\lim_{x\to \infty} cot^{-1} x$

A) 0 B) $\frac{\pi}{2}$ C) π D) $-\frac{\pi}{2}$

Solution:

- Let $\theta = \cot^{-1} x$. Then $\cot\theta = x$.
- As $x \to \infty$, $\cot\theta \to \infty$, which occurs when $\theta \to 0^+$ (since $\cot 0^+$ is very large and positive).
- Therefore, $\lim_{x\to\infty} \cot^{-1} x = 0.$

Answer: A

Find the limit: $\lim_{x\to \infty} \sec^{-1}\left(\frac{x^2+1}{x+1}\right)$

A) 0 B) $\frac{\pi}{3}$ C) $\frac{\pi}{2}$ D) π

Solution:

- Rewrite the expression:

$$\frac{x^2+1}{x+1} = \frac{x^2\left(1+\frac{1}{x^2}\right)}{x\left(1+\frac{1}{x}\right)} = x\cdot\frac{1+\frac{1}{x^2}}{1+\frac{1}{x}}.$$

- As $x \to \infty$, $\frac{1+\frac{1}{x^2}}{1+\frac{1}{x}} \to \frac{1+0}{1+0} = 1$, so $\frac{x^2+1}{x+1} \to \infty$.
- Let $\theta = \sec^{-1} y$. As $y \to \infty$, $\theta \to \frac{\pi}{2}$ (since $\sec\frac{\pi}{2}$ is unbounded).
- Therefore, $\lim_{x\to\infty} \sec^{-1}\left(\frac{x^2+1}{x+1}\right) = \frac{\pi}{2}$

Answer: C

Find the limit: $\lim_{x\to 0} \sin\left(e^{-1/x^2}\right)$

A) 0 B) 1 C) ∞ D) Does not exist

Solution:

- As $x \to 0$, we have $x^2 \to 0^+ \Rightarrow \frac{1}{x^2} \to \infty$.
- So, $-\frac{1}{x^2} \to -\infty$ and hence $e^{-1/x^2} \to 0$.
- Since $\sin t \to \sin 0 = 0$ as $t \to 0$,

$$\lim_{x\to 0} \sin\left(e^{-1/x^2}\right) = 0.$$

Answer: A

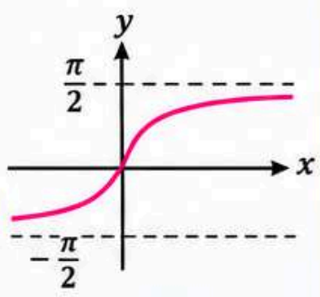
Find the limit: $\lim_{x\to \infty} \sin(\tan^{-1} x)$

A) 0 B) $\frac{1}{2}$ C) 1 D) ∞

Solution:

- As $x \to \infty$, $\tan^{-1} x \to \frac{\pi}{2}$.
- Therefore,

$$\lim_{x\to\infty} \sin(\tan^{-1} x) = \sin\left(\frac{\pi}{2}\right) = 1.$$



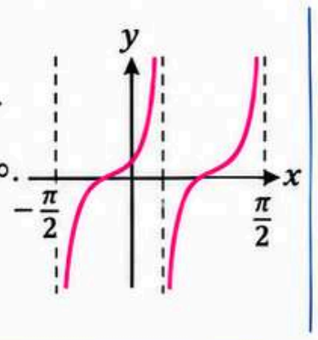
Answer: C

Find the limit: $\lim_{x\to \frac{\pi}{2}} e^{-\tan x}$

A) 0 B) DNE C) ∞ D) $-\infty$

Solution:

- As $x \to \left(\frac{\pi}{2}\right)^-$, $\tan x \to +\infty \Rightarrow e^{-\tan x} \to 0$.
- As $x \to \left(\frac{\pi}{2}\right)^+$, $\tan x \to -\infty \Rightarrow e^{-\tan x} \to \infty$.
- The one-sided limits are different, so the two-sided limit **does not exist.**



Answer: B

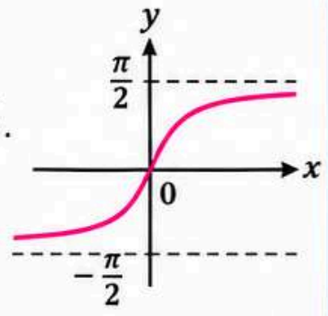
Find the limit: $\lim_{x\to 0^+} \tan^{-1}(\ln x)$

A) 0 B) $\frac{\pi}{2}$ C) $-\frac{\pi}{2}$ D) π

Solution:

- As $x \to 0^+$, $\ln x \to -\infty$.
- So, $\tan^{-1}(\ln x) \to \tan^{-1}(-\infty) = -\frac{\pi}{2}$.
- Therefore,

$$\lim_{x\to 0^+} \tan^{-1}(\ln x) = -\frac{\pi}{2}$$



Answer: C

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Q8	Find the average and instantaneous velocity at a given point	إيجاد السرعة المتوسطة واللحظية عند نقطة معطاة	(19-22)	141	(19-22)	143

Practice Problems

In exercises 19–22, the function represents the position (in feet) of an object at time t seconds. Find the average velocity between (a) $t = 0$ and $t = 2$, (b) $t = 1$ and $t = 2$, (c) $t = 1.9$ and $t = 2$, (d) $t = 1.99$ and $t = 2$, and (e) estimate the instantaneous velocity at $t = 2$.

في التمارين 19–22، تمثل الدالة موقع جسم (بالقدم) عند الزمن t (بالثواني). أوجد السرعة المتوسطة بين (a) $t = 0$ و $t = 2$، (b) $t = 1$ و $t = 2$، (c) $t = 1.9$ و $t = 2$، (d) $t = 1.99$ و $t = 2$، و (e) قدّر السرعة المتجهة اللحظية عند $t = 2$.

Formulas

- **Average velocity:** $v_{avg} = \dfrac{s(t_2) - s(t_1)}{t_2 - t_1}$
- **Instantaneous velocity:** $v(t) = s'(t)$

19 $s(t) = 16t^2 + 10$

Solution:

- $s(t) = 16t^2 + 10$
- $s(0) = 10$, $s(1) = 26$, $s(1.9) = 67.76$, $s(1.99) = 73.3616$, $s(2) = 74$.
- $s'(t) = 32t$.

(a) From $t = 0$ to $t = 2$: $v_{avg} = \dfrac{s(2) - s(0)}{2 - 0} = \dfrac{74 - 10}{2} =$ **32 ft/s**

(b) From $t = 1$ to $t = 2$: $v_{avg} = \dfrac{74 - 26}{2 - 1} =$ **48 ft/s**

(c) From $t = 1.9$ to $t = 2$: $v_{avg} = \dfrac{74 - 67.76}{2 - 1.9} =$ **62.4 ft/s**

(d) From $t = 1.99$ to $t = 2$: $v_{avg} = \dfrac{74 - 73.3616}{2 - 1.99} =$ **63.84 ft/s**

(e) Instantaneous velocity at $t = 2$: $v(2) = s'(2) = 32(2) =$ **64 ft/s**

20 $s(t) = 3t^2 + t$

Solution:

- $s(t) = 3t^2 + t$
- $s(0) = 0$, $s(1) = 4$, $s(1.9) = 12.27$, $s(1.99) = 13.8403$, $s(2) = 14$.
- $s'(t) = 6t + 1$.

(a) From $t = 0$ to $t = 2$: $v_{avg} = \dfrac{s(2) - s(0)}{2 - 0} = \dfrac{14 - 0}{2} =$ **7 ft/s**

(b) From $t = 1$ to $t = 2$: $v_{avg} = \dfrac{14 - 4}{2 - 1} =$ **10 ft/s**

(c) From $t = 1.9$ to $t = 2$: $v_{avg} = \dfrac{14 - 12.27}{2 - 1.9} =$ **17.3 ft/s**

(d) From $t = 1.99$ to $t = 2$: $v_{avg} = \dfrac{14 - 13.8403}{2 - 1.99} =$ **15.97 ft/s**

(e) Instantaneous velocity at $t = 2$: $v(2) = s'(2) = 6(2) + 1 =$ **13 ft/s**

In exercises 19–22, the function represents the position (in feet) of an object at time t seconds. Find the average velocity between (a) $t = 0$ and $t = 2$, (b) $t = 1$ and $t = 2$, (c) $t = 1.9$ and $t = 2$, (d) $t = 1.99$ and $t = 2$, and (e) estimate the instantaneous velocity at $t = 2$.

في التمارين 19–12، تمثل الدالة موقع جسم ما بالقدم عند الزمن t (بالثواني). أوجد السرعة المتجهة المتوسطة بين $t = 2$ و $t = 0$ (a). (b) $t = 2$ و $t = 2$ و $t = 2$ و $t = 2$ و (c) $t = 2$ و $t = 2$ و (d) $t = 1.99$ و (e) قدّر السرعة المتجهة اللحظية عند $t = 2$.

Formulas

- Average velocity: $v_{avg} = \dfrac{s(t_2) - s(t_1)}{t_2 - t_1}$
- Instantaneous velocity: $v(t) = s'(t)$

21 $s(t) = \sqrt{t^2 + 8t}$

Solution:

- $s(t) = \sqrt{t^2 + 8t}$
- $s(0) = 0$, $s(1) = 3$, $s(1.9) = 3.87298$, $s(1.99) = 4.46$, $s(2) = 4$
- $s'(t) = \dfrac{2t + 8}{2\sqrt{t^2 + 8t}}$

(a) From $t = 0$ to $t = 2$: $v_{avg} = \dfrac{s(2) - s(0)}{2 - 0} = \dfrac{4 - 0}{2} =$ **2.24 ft/s**

(b) From $t = 1$ to $t = 2$: $v_{avg} = \dfrac{s(2) - s(1)}{2 - 1} = \dfrac{4 - 3}{1} =$ **1.47 ft/s**

(c) From $t = 1.9$ to $t = 2$: $v_{avg} = \dfrac{4 - 3.87298}{2 - 1.9} =$ **1.35 ft/s**

(d) From $t = 1.99$ to $t = 2$: $v_{avg} = \dfrac{4 - 4.46}{2 - 1.99} =$ **1.34 ft/s**

(e) Instantaneous velocity at $t = 2$: $s'(2) = \dfrac{2(2) + 8}{2\sqrt{(2)^2 + 8(2)}} = \dfrac{12}{2\sqrt{20}} =$ **1.34 ft/s**

22 $s(t) = 3\sin(t - 2)$

Solution:

- $s(t) = 3\sin(t - 2)$
- $s(0) = 3\sin(-2) = -2.7279$
- $s(1) = 3\sin(-1) = -2.5244$
- $s(1.9) = 3\sin(-0.1) = -0.2995$
- $s(1.99) = 3\sin(-0.01) = -0.03$
- $s(2) = 3\sin(0) = 0$
- $s'(t) = 3\cos(t - 2)$

(a) From $t = 0$ to $t = 2$: $v_{avg} = \dfrac{s(2) - s(0)}{2 - 0} = \dfrac{0 - (-2.7279)}{2} =$ **1.3639 ft/s**

(b) From $t = 1$ to $t = 2$: $v_{avg} = \dfrac{0 - (-2.5244)}{2 - 1} =$ **2.5244 ft/s**

(c) From $t = 1.9$ to $t = 2$: $v_{avg} = \dfrac{0 - (-0.2995)}{2 - 1.9} =$ **2.995 ft/s**

(d) From $t = 1.99$ to $t = 2$: $v_{avg} = \dfrac{0 - (-0.03)}{2 - 1.99} =$ **3 ft/s**

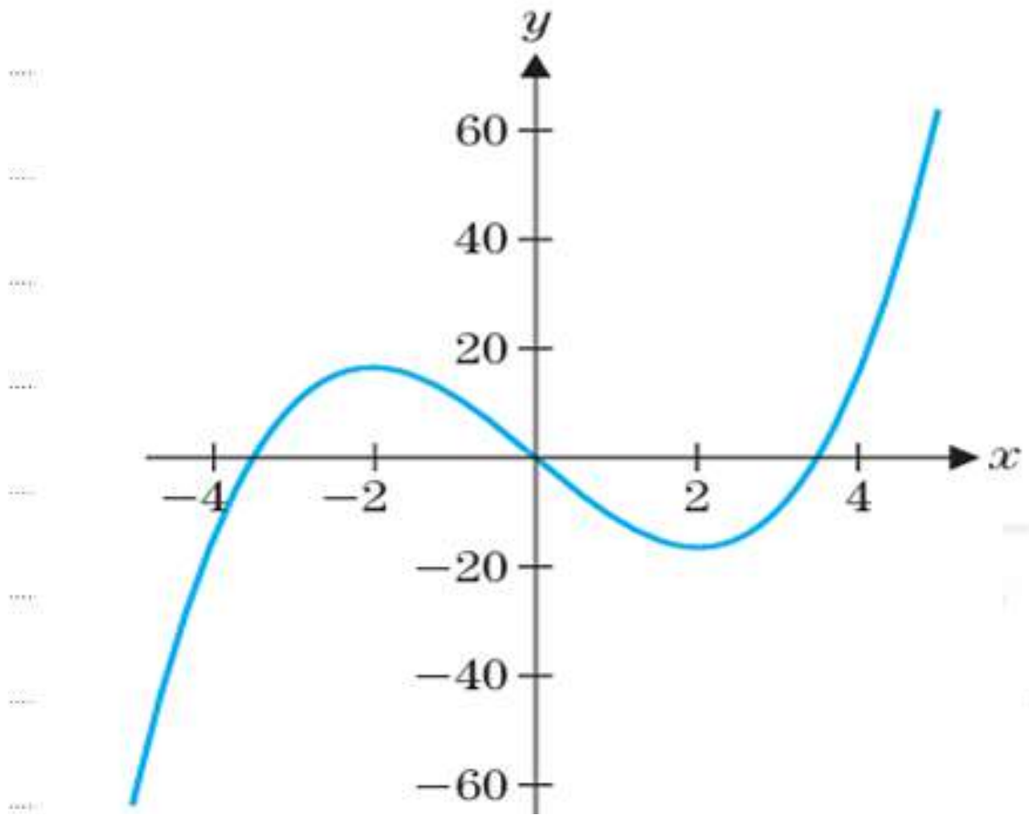
(e) Instantaneous velocity at $t = 2$: $s'(2) = 3\cos(2 - 2) = 3\cos(0) =$ **3 ft/s**

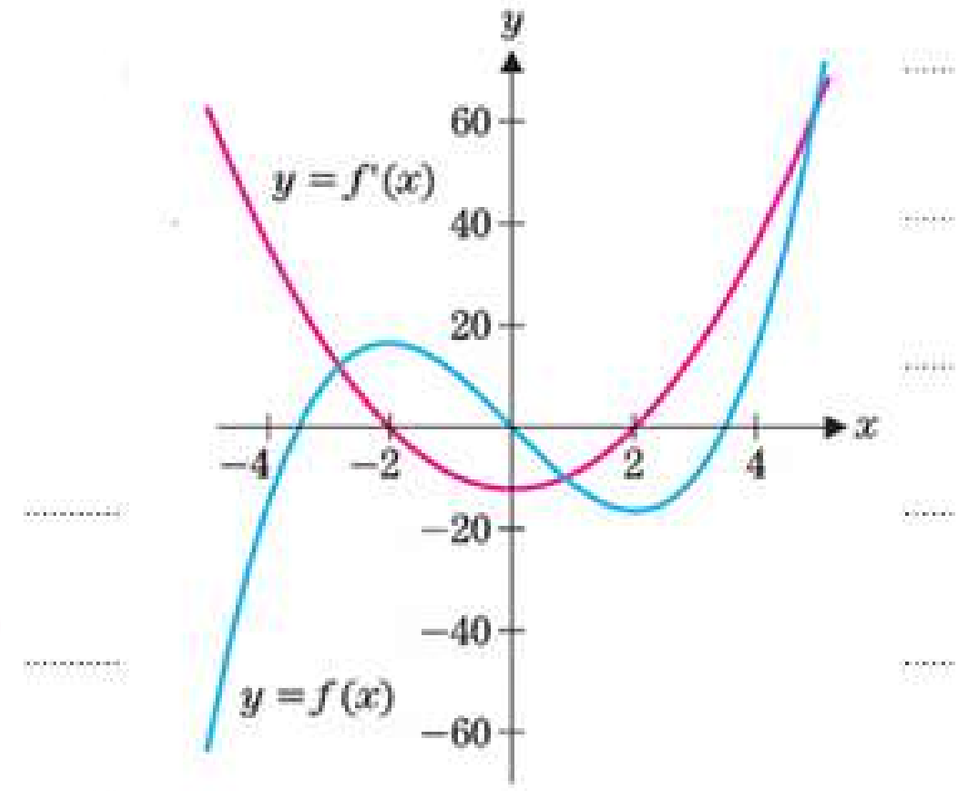
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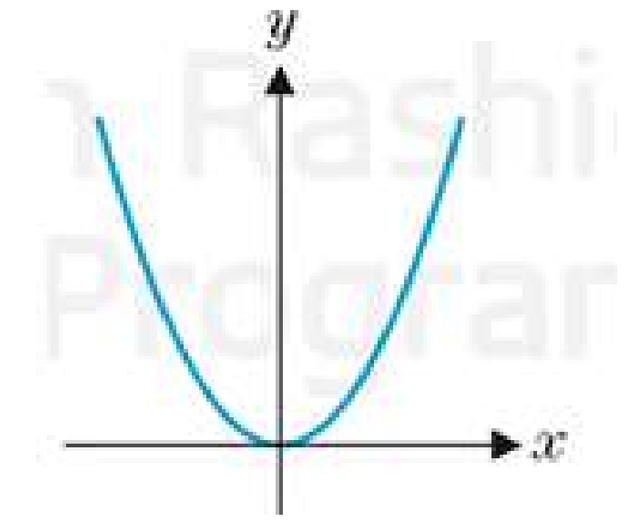
Q9	Sketch the graph of a function using that of its derivative	رسم منحنى دالة اعتماداً على منحنى مشتقتها	Example 2.5	147	Example 2.5	149

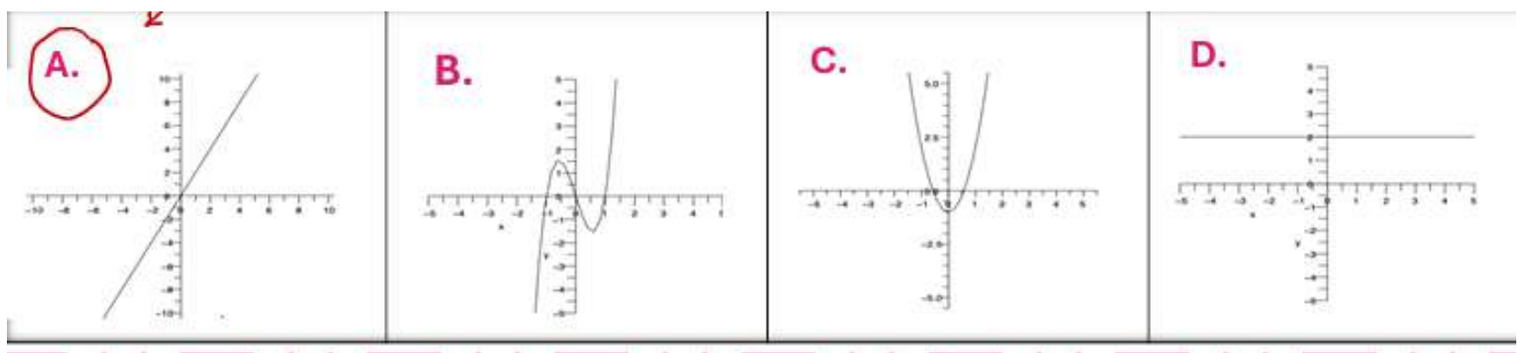
EXAMPLE 2.5 Sketching the Graph of f' Given the Graph of f

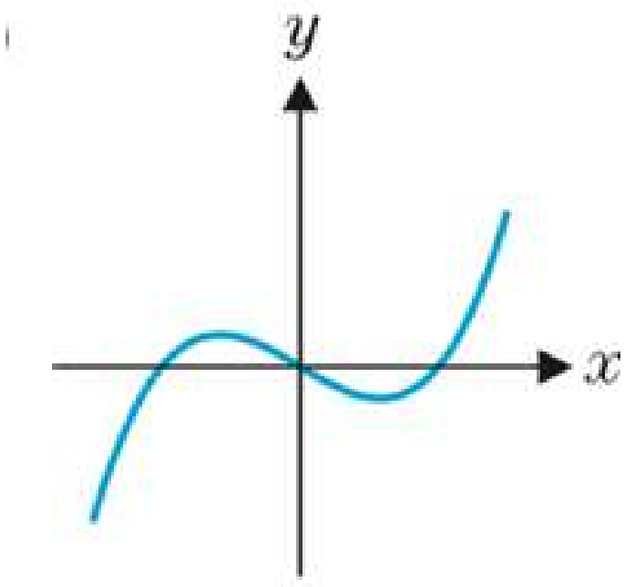
Given the graph of f in Figure 3.14, sketch a plausible graph of f'.

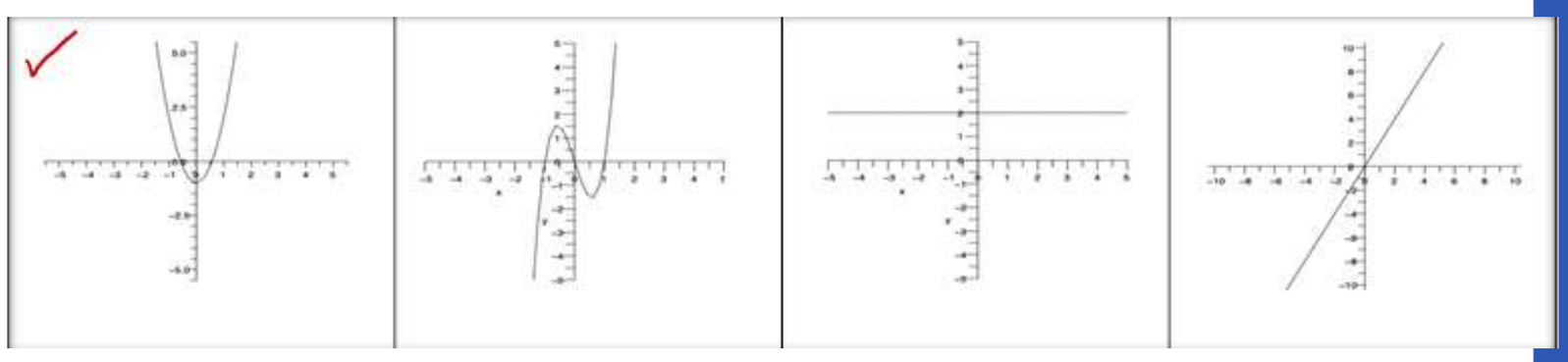


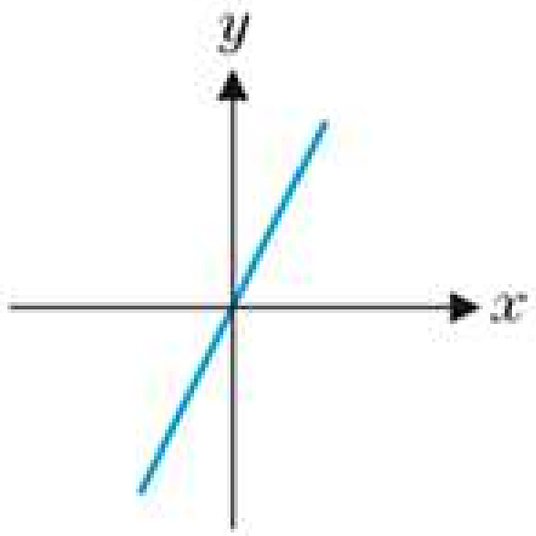


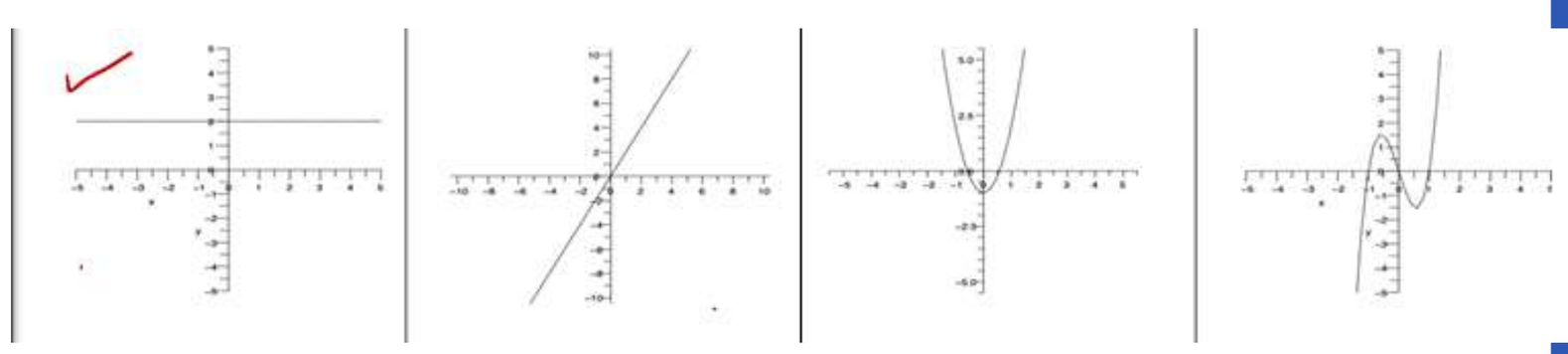








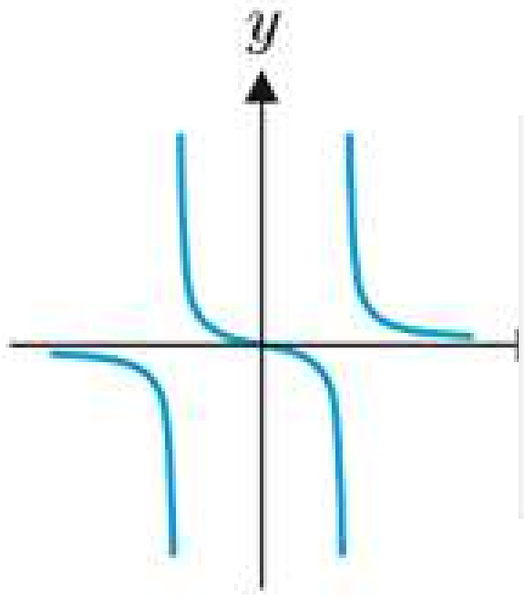


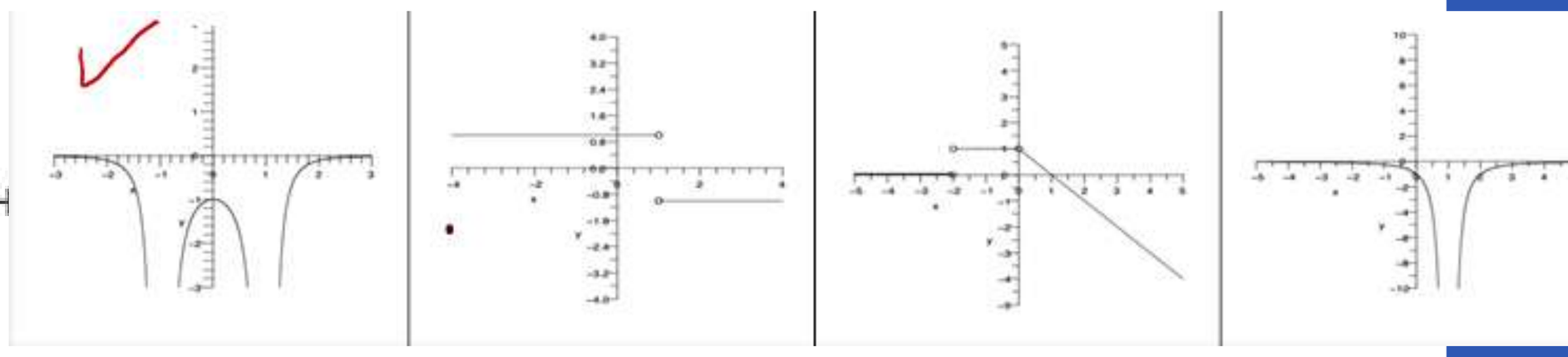


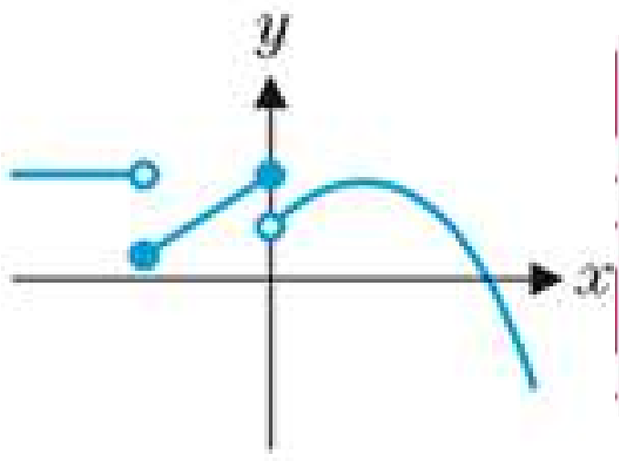
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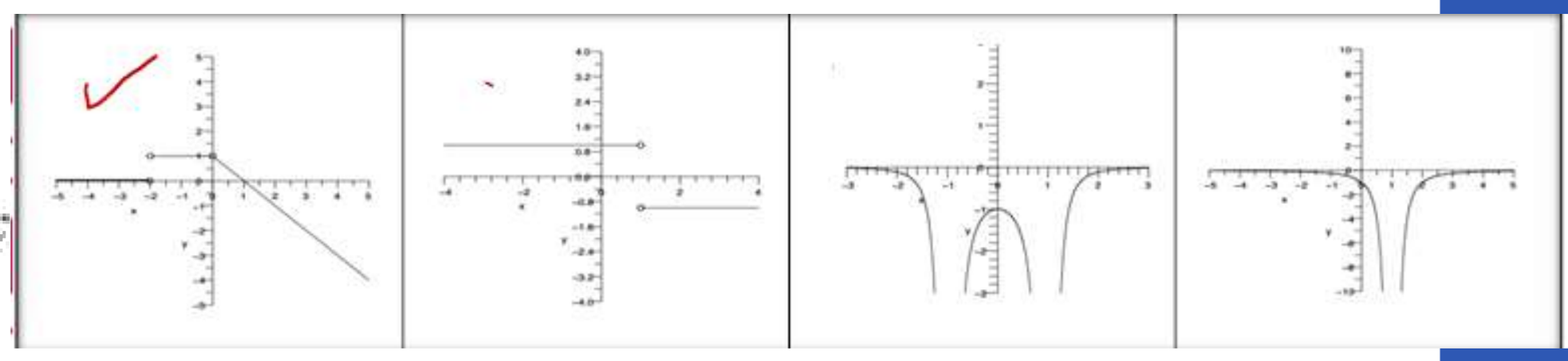


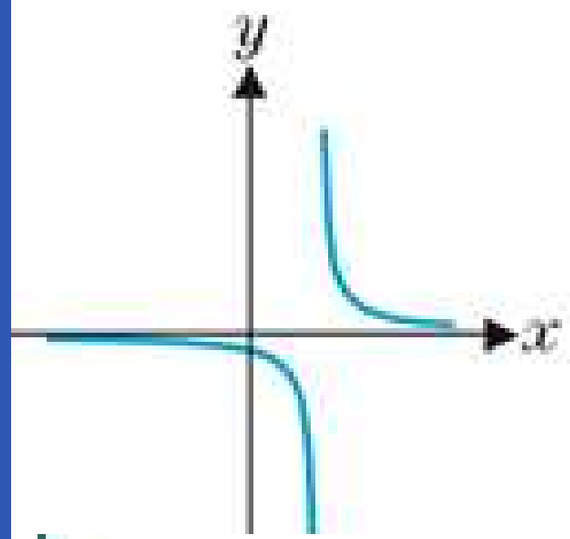


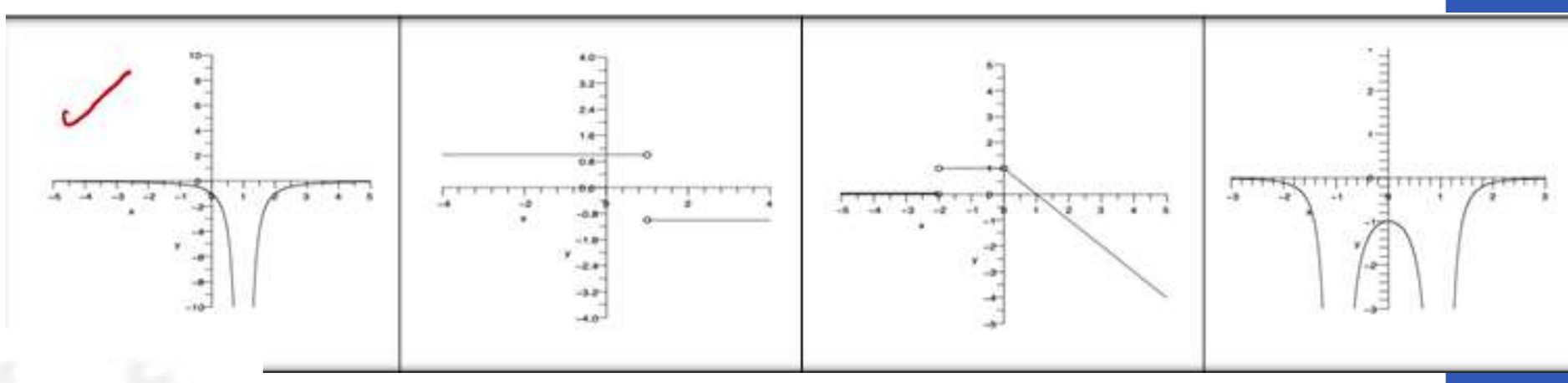


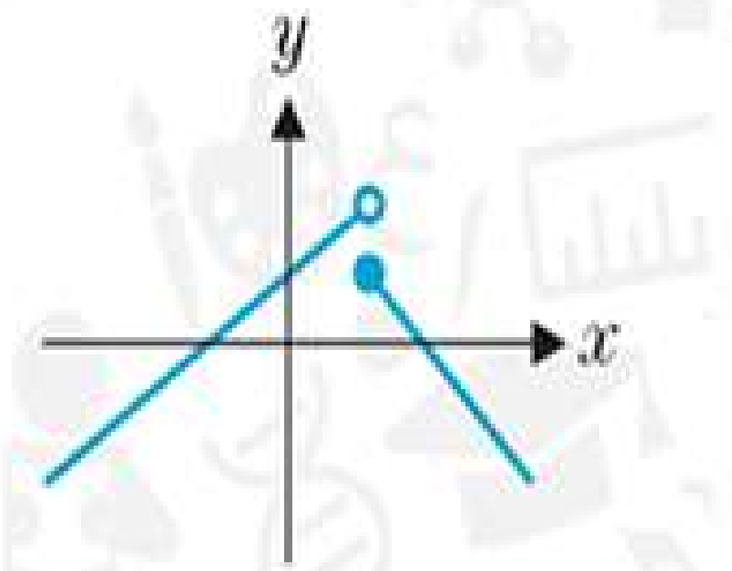




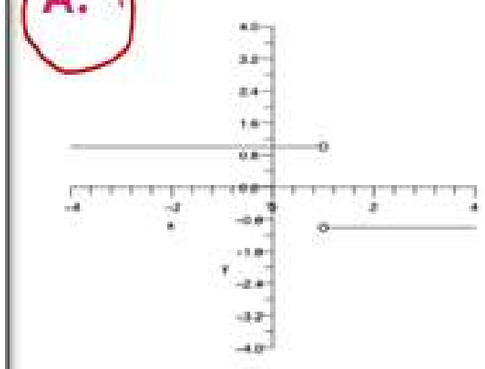




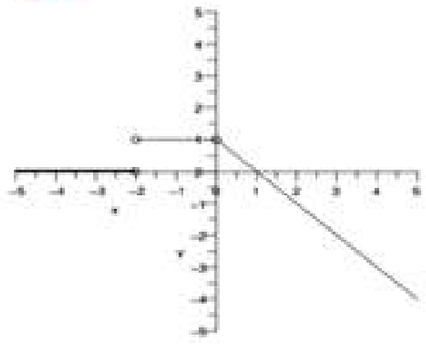




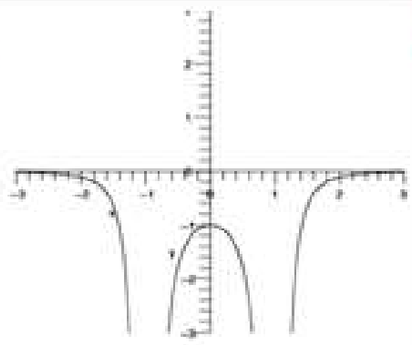
A.



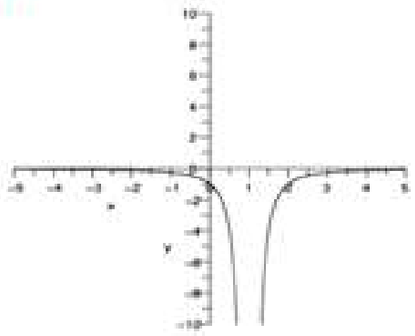
B.



C.



D.

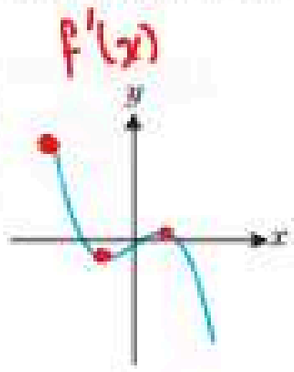


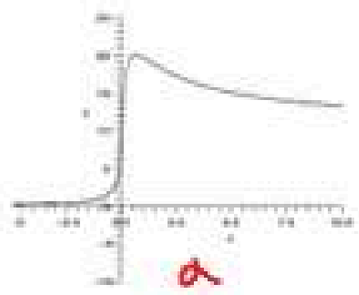
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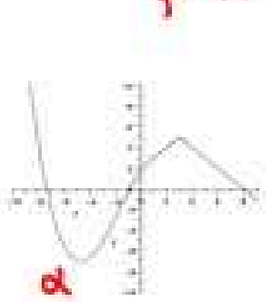
In exercises 17 and 18, use the given graph of f' to sketch a plausible graph of a continuous function f.

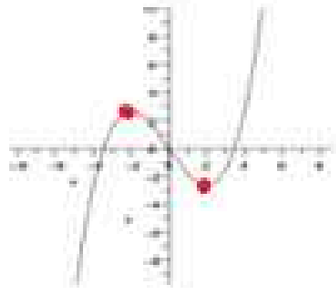
في التمارين 17 و 18، استخدم الرسم البياني المعطى للدالة f' (المشتقة الثانية) لرسم رسم بياني معقول للدالة f المستمرة.

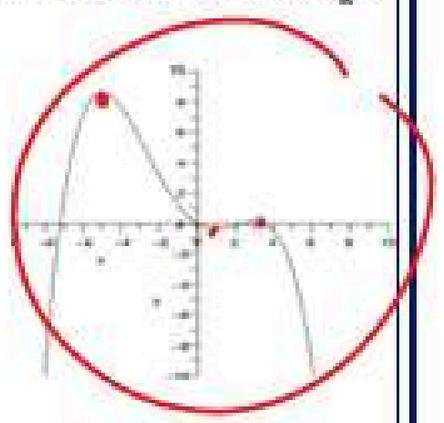
17. (a)



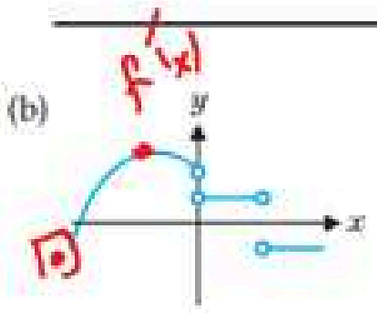


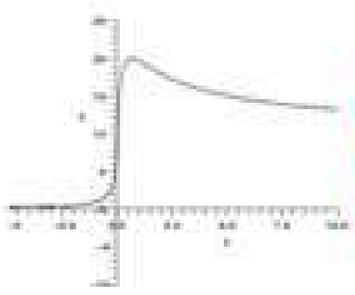


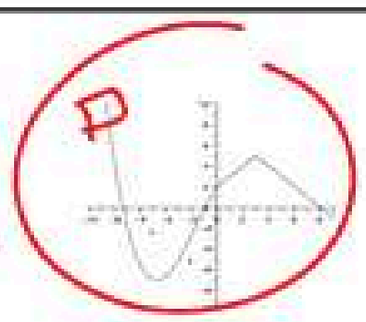


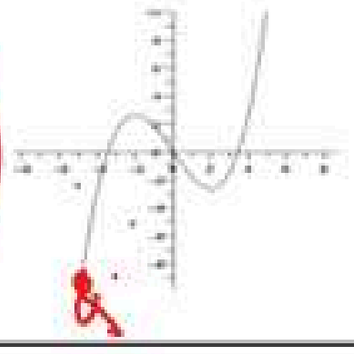


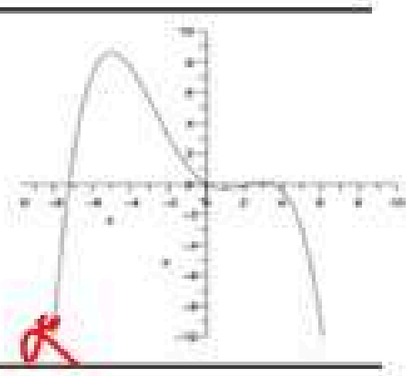
(b)



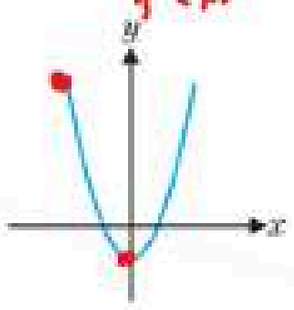


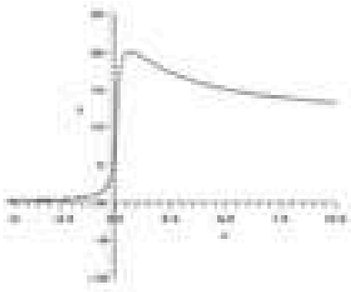


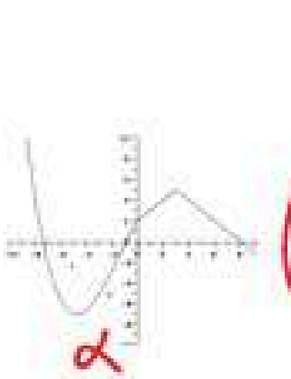


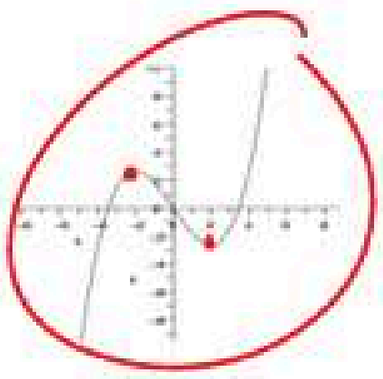


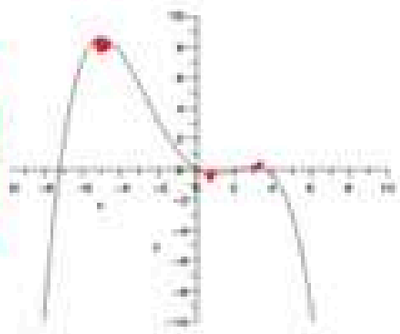
18. (a)



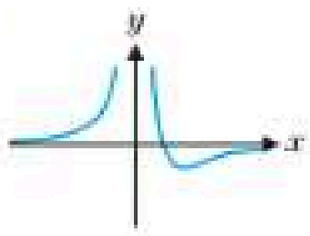


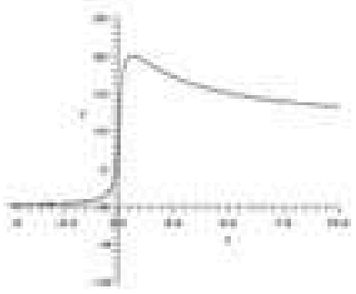


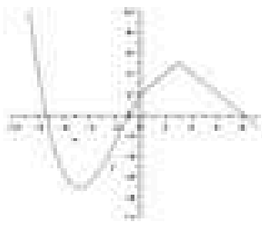


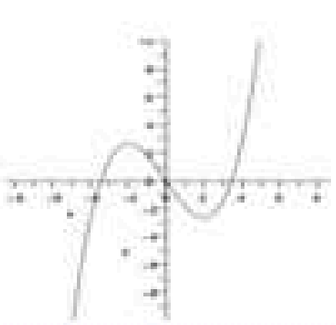


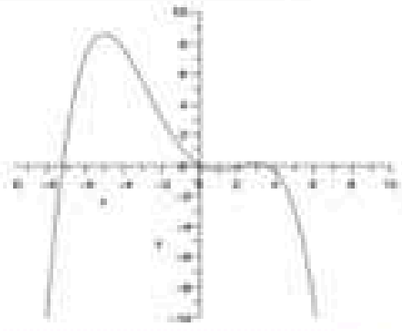
(b)











Q10	**Find the derivative of a function at a given point**	**إيجاد المشتقة للدالة عند نقطة معينة**	(29-30)	152	(29-30)	154

29 (a) $f(x) = |x| + |x - 2|$

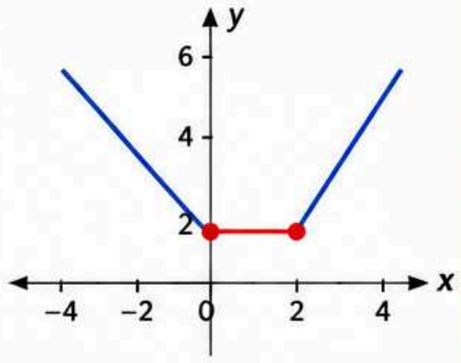
Key steps:

1 Critical points:
$x = 0, 2$

2 Piecewise form:

$$f(x) = \begin{cases} -2x + 2, & x < 0 \\ 2, & 0 \le x \le 2 \\ 2x - 2, & x > 2 \end{cases}$$

x	$f(x)$
−1	4
0	2
1	2
2	2
3	4



Not differentiable at: $x = 0, 2$

29 (b) $f(x) = |x^2 - 4x|$

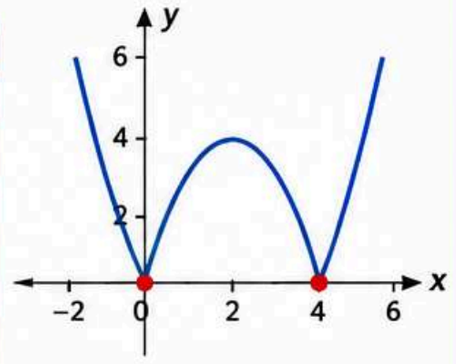
Key steps:

1 Critical points:
$x^2 - 4x = 0 \Rightarrow x = 0, 4$

2 Piecewise form:

$$f(x) = \begin{cases} x^2 - 4x, & x \le 0 \text{ or } x \ge 4 \\ 4x - x^2, & 0 \le x \le 4 \end{cases}$$

x	$f(x)$
−1	5
0	0
1	3
2	4
3	3
4	0



Not differentiable at: $x = 0, 4$

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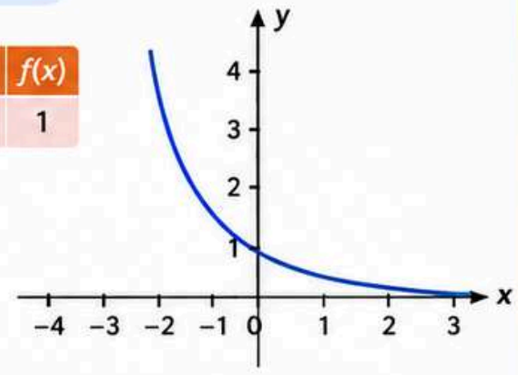


30 (a) $f(x) = e^{-2x}$

Key steps:

1. **Critical points:** None (e^{-2x} is differentiable for all real x).
2. **Behavior:** A smooth exponential decay curve.

x	$f(x)$
0	1



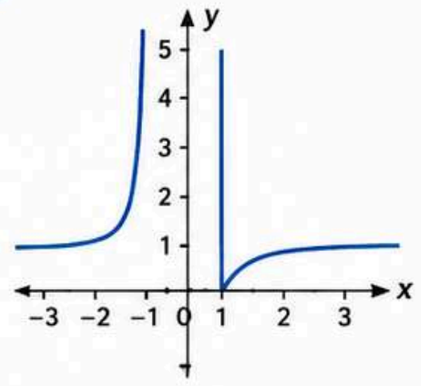
Not differentiable at: None
(It is differentiable for all real x).

30 (b) $g(x) = e^{-2/(x^2-x)}$

Key steps:

1. **Domain:** $x^2 - x \neq 0 \Rightarrow x(x-1) \neq 0$ $\Rightarrow x \neq 0, 1$
2. **Behavior:** The function is not defined at $x = 0$ and $x = 1$, so it is not differentiable there.

x	$g(x)$
−1	error
0	error
1	error



Not differentiable at: $x = 0, 1$

EXAM!

Q1 Which of the following functions is not differentiable at $x = 2$?

A $f(x) = \begin{cases} 4 & , \ x < 2 \\ 2x & , \ x \geq 2 \end{cases}$

B $g(x) = \dfrac{2}{x+2}$

C $h(x) = |x-2|^2$

D $Q(x) = x^2 - 4$

Solution

- $f(x) = \begin{cases} 4 & , \ x < 2 \\ 2x & , \ x \geq 2 \end{cases}$.
- At $x = 2$: $f(2^-) = 4$, $f(2^+) = 2(2) = 4$ $\Rightarrow$ continuous at $x = 2$.
- Left derivative: $f'_-(2) = 0$ (derivative of constant 4).
- Right derivative: $f'_+(2) = 2$ (derivative of $2x$).
- Since $f'_-(2) \neq f'_+(2)$, f is **not differentiable** at $x = 2$.
- $g(x)$, $h(x)$, $Q(x)$ are differentiable at $x = 2$ (rational, polynomial, and square of an absolute value).

Answer: A

Q2 Which of the following functions is not differentiable at $x = 2$?

A $f(x) = |x-2|^2$

B $g(x) = x^2 - 4$

C $p(x) = \begin{cases} x^2 + 1 & , \ x \geq 2 \\ 4x - 1 & , \ x < 2 \end{cases}$

D $h(x) = \dfrac{1}{x+2}$

Solution

- $p(x) = \begin{cases} x^2 + 1 & , \ x \geq 2 \\ 4x - 1 & , \ x < 2 \end{cases}$.
- At $x = 2$: $p(2^-) = 4(2) - 1 = 7$, $p(2^+) = 2^2 + 1 = 5$
- Since $p(2^-) \neq p(2^+)$, p is **not continuous** at $x = 2$, and hence **not differentiable** at $x = 2$.
- Check the others:
  - $f(x) = (x-2)^2$: differentiable at $x = 2$.
  - $g(x) = x^2 - 4$: differentiable at $x = 2$.
  - $h(x) = \dfrac{1}{x+2}$: differentiable at $x = 2$ (since $2 \neq -2$).

Answer: C

Q11	Find the derivative of a function using the Power Rule at a given point	أوجد مشتقة دالة باستخدام قاعدة القوة عند نقطة معينة	(7-14)	160	(7-14)	162

Q7 $h(x) = \dfrac{10}{\sqrt[3]{x}} - 2x + \pi$

Solution

$h(x) = 10x^{-\frac{1}{3}} - 2x + \pi$

$h'(x) = 10\left(-\frac{1}{3}\right)x^{-4/3} - 2$

$h'(x) = -\dfrac{10}{3x^{4/3}} - 2$

A $h'(x) = -\dfrac{10}{3x^{4/3}} - 2$

B $h'(x) = -\dfrac{10}{3x^3\sqrt[3]{x}}$

C $h'(x) = \dfrac{10}{3x^3\sqrt[3]{x}} - 2$

D $h'(x) = -\dfrac{10}{\sqrt[3]{x}} - 2$

Q8 $h(x) = 12x - x^2 - \dfrac{3}{\sqrt{x}}$

Solution

$h(x) = 12x - x^2 - 3x^{-\frac{1}{2}}$

$h'(x) = 12 - 2x - 3\left(-\frac{1}{2}\right)x^{-3\frac{3}{2}}$

$h'(x) = 12 - 2x + \dfrac{3}{2x^{3/2}}$

A $h'(x) = 12 - 2x + \dfrac{3}{x^{3/2}}$

B $h'(x) = 12 - 2x - \dfrac{2}{x}$

C $h'(x) = -2x - \dfrac{2}{x^3\sqrt[4]{x}}$

D $h'(x) = 12 - 2x + \dfrac{3}{2x^{3/2}}$

Q9 $f(s) = 2s^{3/2} - 3s^{-1/3}$

Solution

$f'(s) = 2\left(\frac{3}{2}\right)s^{1/2} - 3\left(-\frac{1}{3}\right)s^{-4\frac{4}{3}}$

$f'(s) = 3s^{1/2} + s^{-4\frac{4}{3}}$

$f'(s) = 3\sqrt{s} + \dfrac{1}{s^{4/3}} = 3\sqrt{s} + \dfrac{1}{\sqrt[3]{s^4}}$

A $f'(s) = 3s^{1/2} + s^{-4/3}$

B $f'(s) = 3 + \dfrac{1}{\sqrt[3]{s^4}}$

C $f'(s) = 3\sqrt{s} - \dfrac{1}{\sqrt[3]{s^4}}$

D $f'(s) = \sqrt{s} + \dfrac{1}{\sqrt[3]{s^4}}$

Q10 $f(t) = 3t^4 - 2t^{1/3}$

Solution

$f'(t) = 3(4)t^3 - 2\left(\frac{1}{3}\right)t^{-2\frac{2}{3}}$

$f'(t) = 12t^3 - \dfrac{2}{3}t^{-2/3}$

$f'(t) = 12t^3 - \dfrac{2}{3t^{2/3}}$

A $f'(t) = 3\pi t^{-1} - 2.6t^3$

B $f'(t) = 3\pi t^{-1} + 2.6t^{0.3}$

C $f'(t) = \pi t^{-1} - 2.6t^{0.3}$

D $f'(t) = 12t^3 - \dfrac{2}{3t^{2/3}}$

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Q11 $f(x) = \dfrac{3x^2 - 3x + 1}{2x}$

A $f'(x) = \dfrac{3}{2} - \dfrac{1}{2x^2}$

B $f'(x) = \dfrac{3}{2}x - \dfrac{1}{2x^2}$

C $f'(x) = -\dfrac{3}{2} + \dfrac{1}{2x^2}$

D $f'(x) = \dfrac{3}{2} - \dfrac{7}{2x^3}$

Solution

$$f(x) = \frac{3x^2 - 3x + 1}{2x} = \frac{3}{2}x - \frac{3}{2} + \frac{1}{2}x^{-1}$$

$$f'(x) = \frac{3}{2} - \frac{1}{2}x^{-2} = \frac{3}{2} - \frac{1}{2x^2}$$

Answer: A

Q12 $f(x) = \dfrac{4x^2 - x + 3}{\sqrt{x}}$

A $f'(x) = 6x^{\frac{1}{2}} - \frac{1}{2}x^{-\frac{1}{2}} - \frac{3}{2}x^{-\frac{3}{2}}$

B $f'(x) = -\frac{1}{2}x^{-\frac{1}{2}} - \frac{3}{2}x^{-\frac{3}{2}}$

C $f'(x) = 6x^{\frac{1}{2}} - \frac{1}{2}x^{-\frac{1}{2}} + \frac{3}{2}x^{-\frac{3}{2}}$

D $f'(x) = 6x^{\frac{1}{2}} - \frac{1}{2}x^{-\frac{1}{2}}$

Solution

$$f(x) = \frac{4x^2 - x + 3}{x^{1/2}} = 4x^{\frac{3}{2}} - x^{\frac{1}{2}} + 3x^{-\frac{1}{2}}$$

$$f'(x) = 6x^{\frac{1}{2}} - \frac{1}{2}x^{-\frac{1}{2}} - \frac{3}{2}x^{-\frac{3}{2}}$$

Answer: A

Q13 $f(x) = x(3x^2 - \sqrt{x})$

A $f'(x) = 9x^2 - \frac{3}{2}x^{\frac{1}{2}}$

B $f'(x) = 9x^3 + \frac{3}{2}x^{-\frac{1}{2}}$

C $f'(x) = -9x^2 - \frac{3}{2}x^{\frac{1}{2}}$

D $f'(x) = x^2 - \frac{3}{7}x^{-\frac{1}{2}}$

Solution

$$f(x) = x\left(3x^2 - x^{\frac{1}{2}}\right) = 3x^3 - x^{\frac{3}{2}}$$

$$f'(x) = 9x^2 - \frac{3}{2}x^{\frac{1}{2}}$$

Answer: A

Q14 $f(x) = (x + 1)(3x^2 - 4)$

A $f'(x) = 9x^2 + 6x - 4$

B $f'(x) = 9x^2 - 6x - 16$

C $f'(x) = 9x^4 - x - 4$

D $f'(x) = 9x^3 - 6x - 4$

Solution

$$f(x) = 3x^3 + 3x^2 - 4x - 4$$

$$f'(x) = 9x^2 + 6x - 4$$

Answer: A

Badr Academy — Previous Exam — Calculus

Q1 The function $s(t) = 3t^3 + t$ represents the position in feet of an object at time t seconds. Find the average velocity between $t = 0$ and $t = 2$.

A $v_{avg} = 24\ (ft/s)$

B $v_{avg} = 26\ (ft/s)$

C $v_{avg} = 13\ (ft/s)$

D $v_{avg} = 12\ (ft/s)$

Solution

- Average velocity: $v_{avg} = \dfrac{s(2) - s(0)}{2 - 0}$.
- $s(t) = 3t^3 + t \Rightarrow s(2) = 3(2)^3 + 2 = 26,\ s(0) = 0.$
- So, $v_{avg} = \dfrac{26 - 0}{2} = 13\ (ft/s).$

Answer: C

Q2 A ball is thrown upward then moves according to the relation $s(t) = 56t - 4t^2$, where t is in seconds and s in meters. What is the maximum height the ball can reach?

A $7\ m$

B $168\ m$

C $196\ m$

D $392\ m$

Solution

- Maximum height occurs when $v(t) = s'(t) = 0$.
- $s(t) = 56t - 4t^2 \Rightarrow v(t) = 56 - 8t.$
- Set $56 - 8t = 0 \Rightarrow t = 7\ s.$
- $s(7) = 56(7) - 4(7)^2 = 196\ m.$

Answer: C

Q3 The function $s(t) = 2\sqrt{t} + 2t^2$ represents the position of an object. Find the acceleration function.

A $a(t) = \frac{1}{2}t^{3/2} + 4$

B $a(t) = -\frac{1}{2}t^{-3/2} + 4$

C $a(t) = t^{-1/2} + 4t$

D $a(t) = t^{1/2} + 4t$

Solution

- $s(t) = 2\sqrt{t} + 2t^2 = 2t^{\frac{1}{2}} + 2t^2.$
- $v(t) = s'(t) = 2\left(\frac{1}{2}\right)t^{-\frac{1}{2}} + 4t = t^{-\frac{1}{2}} + 4t.$
- $a(t) = v'(t) = \frac{1}{2}t^{-\frac{3}{2}} + 4.$

Answer: B

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Q3 The function $s(t) = 2\sqrt{t} + 2t^2$ represents the position of an object. Find the acceleration function.

- **A** $a(t) = \frac{1}{2}t^{3/2} + 4$
- **B** $a(t) = -\frac{1}{2}t^{-3/2} + 4$
- **C** $a(t) = t^{-1/2} + 4t$
- **D** $a(t) = t^{1/2} + 4t$

Solution

- $s(t) = 2\sqrt{t} + 2t^2 = 2t^{\frac{1}{2}} + 2t^2.$
- $v(t) = s'(t) = 2\left(\frac{1}{2}\right)t^{-\frac{1}{2}} + 4t = t^{-\frac{1}{2}} + 4t.$
- $a(t) = v'(t) = \frac{1}{2}t^{-\frac{3}{2}} + 4.$

Answer: B

Q4 Use the given position function $s(t) = \sqrt{t} + 2t^2$ to find the acceleration at $t = 1$.

- **A** $-\frac{15}{2}$
- **B** $\frac{15}{2}$
- **C** $\frac{15}{4}$
- **D** $-\frac{15}{4}$

Solution

- $s(t) = t^{\frac{1}{2}} + 2t^2.$
- $v(t) = s'(t) = \frac{1}{2}t^{-\frac{1}{2}} + 4t.$
- $a(t) = v'(t) = \frac{1}{4}t^{-\frac{3}{3}} + 4.$
- At $t = 1$: $a(1) = \frac{1}{4} + 4 = \frac{15}{2}.$

Answer: B

Q1 Find the derivative of the function $f(t) = 3t^{\pi} - 2t^{1.3}$.

- **A** $f'(t) = 3\pi - 2.6t$
- **B** $f'(t) = 3\pi t - 2.6t$
- **C** $f'(t) = 3\pi t^{\pi-1} - 2.6t^{0.3}$
- **D** $f'(t) = 3\pi t^{\pi^2} - 2.6t^{0.3}$

Solution

- $f(t) = 3t^{\pi} - 2t^{1.3}$
- $f'(t) = 3\pi t^{\pi-1} - 2(1.3)t^{1.3-1}$
- $f'(t) = 3\pi t^{\pi-1} - 2.6t^{0.3}$

Answer: C

Q2 Suppose that the height of a falling object t seconds after being dropped from a height of 64 ft is given by $s(t) = 64 - 26t^2$ ft. Find the average velocity between $t = 1$ and $t = 2$.

- **A** $v_{avg} = -24\ ft/s$
- **B** $v_{avg} = -6\ ft/s$
- **C** $v_{avg} = -36\ ft/s$
- **D** $v_{avg} = -48\ ft/s$

Solution

- Average velocity: $v_{avg} = \frac{s(2) - s(1)}{2 - 1}.$
- $s(1) = 64 - 26(1)^2 = 38.$
- $s(2) = 64 - 26(2)^2 = 64 - 104 = -40.$
- $v_{avg} = \frac{-40 - 38}{1} = -78\ ft/s.$

Answer: C

Q3 The function $h(t) = 10t^2 - 24t$ represents the height of an object. Compute the velocity and acceleration at time $t = 1$.

- **A** $v(1) = -4,\ a(1) = -20$
- **B** $v(1) = -4,\ a(1) = 20$
- **C** $v(1) = 4,\ a(1) = -20$
- **D** $v(1) = 4,\ a(1) = 20$

Solution

- $h(t) = 10t^2 - 24t.$
- $v(t) = h'(t) = 20t - 24.$
- $a(t) = h''(t) = 20.$
- $v(1) = 20(1) - 24 = -4.$
- $a(1) = 20.$

Answer: B

Q4 If $f(x) = x^4 - 5x$ then $\lim_{x\to 2} \frac{f'(x) - f'(2)}{x - 2}$ equals

- **A** 6
- **B** 27
- **C** 38
- **D** 48

Solution

- $f(x) = x^4 - 5x \Rightarrow f'(x) = 4x^3 - 5.$
- $f'(2) = 4(2)^3 - 5 = 27.$
- $\lim_{x\to 2} \frac{f'(x) - f'(2)}{x - 2} = \lim_{x\to 2} \frac{4x^3 - 5 - 27}{x - 2}$
  $= \lim_{x\to 2} \frac{4(x^3 - 8)}{x - 2} = \lim_{x\to 2} \frac{4(x - 2)(x^2 + 2x + 4)}{x - 2}$
  $= 4(2^2 + 2\cdot 2 + 4) = 4(4 + 4 + 4) = 48.$

Answer: D

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In exercises 21–24, assume that f and g are differentiable with
$f(0) = -1,\ f(1) = -2,\ f'(0) = -1,\ f'(1) = 3,\ g(0) = 3,\ g(1) = 1,\ g'(0) = -1,\ g'(1) = -2.$
Find the equation of the tangent line to the graph of $y = h(x)$ at $x = a$, where $h(x) = f(x)g(x)$.

Q21 (a) $h(x) = f(x)g(x);\ a = 0$

Solution

- $h(0) = f(0)g(0) = (-1)(3) = -3.$
- $h'(x) = f'(x)g(x) + f(x)g'(x).$
- $h'(0) = f'(0)g(0) + f(0)g'(0) = (-1)(3) + (-1)(-1)$
  $= -3 + 1 = -2.$

Tangent line at $x = a$:
$y - y_1 = m(x - x_1),$
where $m = h'(a)$ and $y_1 = h(a).$

- $y - y_1 = m(x - x_1) \Rightarrow y + 3 = -2(x - 0).$
- $y + 3 = -2x \Rightarrow$ **$y = -2x - 3.$**

Q21 (b) $h(x) = f(x)g(x);\ a = 1$

Solution

- $h(1) = f(1)g(1) = (-2)(1) = -2.$
- $h'(x) = f'(x)g(x) + f(x)g'(x).$
- $h'(1) = f'(1)g(1) + f(1)g'(1) = (3)(1) + (-2)(-2)$
  $= 3 + 4 = 7.$

Tangent line at $x = a$:
$y - y_1 = m(x - x_1),$
where $m = h'(a)$ and $y_1 = h(a).$

- $y - y_1 = m(x - x_1) \Rightarrow y + 2 = 7(x - 1).$
- $y + 2 = 7x - 7 \Rightarrow$ **$y = 7x - 9$**

In exercises 21–24, assume that f and g are differentiable with
$f(0) = -1,\ f(1) = -2,\ f'(0) = -1,\ f'(1) = 3,\ g(0) = 3,\ g(1) = 1,$
$g'(0) = -1$ and $g'(1) = -2.$ Find an equation of the tangent line to the graph of $y = h(x)$ at $x = a.$

Q22 (a) $h(x) = \dfrac{f(x)}{g(x)};\ a = 1$

Solution

Quotient Rule:
$$h'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$$

- $h(1) = \dfrac{f(1)}{g(1)} = \dfrac{-2}{1} = -2.$
- $h'(1) = \dfrac{f'(1)g(1) - f(1)g'(1)}{(g(1))^2}$
  $= \dfrac{(3)(1) - (-2)(-2)}{(1)^2} = 3 - 4 = -1.$
- **Tangent line at $x = 1$:**
  $y - h(1) = h'(1)(x - 1) \Rightarrow y + 2 = -1(x - 1)$
  $\Rightarrow$ **$y = -x - 1.$**

Q22 (b) $h(x) = \dfrac{f(x)}{g(x)};\ a = 0$

Solution

Quotient Rule:
$$h'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$$

- $h(0) = \dfrac{f(0)}{g(0)} = \dfrac{-1}{3} = -\dfrac{1}{3}.$
- $h'(0) = \dfrac{f'(0)g(0) - f(0)g'(0)}{(g(0))^2}$
  $= \dfrac{(-1)(3) - (-1)(-1)}{(3)^2} = \dfrac{-3 - 1}{9} = \dfrac{-4}{9}.$
- **Tangent line at $x = 0$:**
  $y - h(0) = h'(0)(x - 0) \Rightarrow y + \dfrac{1}{3} = -\dfrac{4}{9}x$
  $\Rightarrow$ **$y = -\dfrac{4}{9}x - \dfrac{1}{3}.$**

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In exercises 21–24, assume that f and g are differentiable with

$f(0) = -1,\ f(1) = -2,\ f'(0) = -1,\ f'(1) = 3,\ g(0) = 3,\ g(1) = 1,\ g'(0) = -1,\ g'(1) = -2.$

Find the equation of the tangent line to the graph of $y = h(x)$ at $x = a$.

Q23 (a)

$h(x) = x^2 f(x);\quad a = 1$

Solution

Product Rule:

$$(uv)' = u'v + uv'$$

Here, $u = x^2,\ u' = 2x,\ v = f(x),\ v' = f'(x).$

- $h(1) = 1^2 f(1) = (1)(-2) = -2.$
- $h'(x) = 2xf(x) + x^2 f'(x).$
- $h'(1) = 2(1)f(1) + (1)^2 f'(1)$
  $= 2(-2) + (1)(3) = -4 + 3 = -1.$
- **Tangent line at $x = 1$:**
  $y - h(1) = h'(1)(x - 1) \Rightarrow y + 2 = -1(x - 1)$
  $\Rightarrow$ **$y = -x - 1.$**

Q23 (b)

$h(x) = x^2 f(x);\quad a = 0$

Solution

Product Rule:

$$(uv)' = u'v + uv'$$

Here, $u = x^2,\ u' = 2x,\ v = f(x),\ v' = f'(x).$

- $h(0) = 0^2 f(0) = 0(-1) = 0.$
- $h'(x) = 2xf(x) + x^2 f'(x).$
- $h'(0) = 2(0)f(0) + (0)^2 f'(0) = 0(-1) + 0(-1) = 0.$
- **Tangent line at $x = 0$:**
  $y - h(0) = h'(0)(x - 0) \Rightarrow y = 0(x) \Rightarrow$ **$y = 0.$**

Q24 (a)

$h(x) = \dfrac{x^2}{g(x)};\quad a = 1$

Solution

Quotient Rule:

$$h'(x) = \frac{2xg(x) - x^2 g'(x)}{(g(x))^2}$$

- $h(1) = \dfrac{1^2}{g(1)} = \dfrac{1}{1} = 1.$
- $h'(x) = \dfrac{2xg(x) - x^2 g'(x)}{(g(x))^2}.$
- $h'(1) = \dfrac{2(1)g(1) - 1^2 g'(1)}{(g(1))^2} = \dfrac{2(1)(1) - (1)(-2)}{(1)^2} = 4.$
- **Tangent line at $x = 1$:**
  $y - h(1) = h'(1)(x - 1) \Rightarrow y - 1 = 4(x - 1)$
  $\Rightarrow y = 4x - 4 + 1 \Rightarrow$ **$y = 4x - 3.$**

Q24 (b)

$h(x) = \dfrac{x^2}{g(x)};\quad a = 0$

Solution

Quotient Rule:

$$h'(x) = \frac{2xg(x) - x^2 g'(x)}{(g(x))^2}$$

- $h(0) = \dfrac{0^2}{g(0)} = \dfrac{0}{3} = 0.$
- $h'(0) = \dfrac{2(0)g(0) - 0^2 g'(0)}{(g(0))^2} = \dfrac{0(3) - 0^2(-1)}{(3)^2} = 0.$
- **Tangent line at $x = 0$:**
  $y - h(0) = h'(0)(x - 0) \Rightarrow y - 0 = 0(x - 0)$
  $\Rightarrow$ **$y = 0.$**

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Find $g'(1)$ if $g(x) = \dfrac{4x^2 - x - 3}{\sqrt{x}}$.

Solution

$$g(x) = 4x^2 - x - 3\,x^{-\frac{1}{2}} = 4x^{\frac{3}{2}} - x^{\frac{1}{2}} - 3x^{-\frac{1}{2}}$$
$$g'(x) = 6x^{\frac{1}{2}} - \frac{1}{2}x^{-\frac{1}{2}} + \frac{3}{2}x^{-\frac{3}{2}}.$$
$$g'(1) = 6(1) - \frac{1}{2}(1) + \frac{3}{2}(1) = \boxed{g'(1) = 7.}$$

If $f(x) = 2x^4 + bx + 3$ and $\lim\limits_{h\to 0}\dfrac{f(2+h)-f(2)}{h} = 10$, find the value of b.

Solution

$$\lim_{h\to 0}\frac{f(2+h)-f(2)}{h} = f'(2) = 10.$$
$$f'(x) = 8x^3 + b \Rightarrow f'(2) = 8(2)^3 + b = 64 + b.$$
$$64 + b = 10 \Rightarrow b = 10 - 64 = \boxed{-54.}$$

If $f(x) = x^4 + 3x^2 - 2$, find $f'''\left(\frac{1}{6}\right)$.

- ○ $f'''\left(\frac{1}{6}\right) = 2$
- ○ $f'''\left(\frac{1}{6}\right) = -4$
- ☑ $f'''\left(\frac{1}{6}\right) = 4$
- ○ $f'''\left(\frac{1}{6}\right) = -2$

Solution

$$f'(x) = 4x^3 + 6x,$$
$$f''(x) = 12x^2 + 6,$$
$$f'''(x) = 24x.$$
$$f'''\left(\frac{1}{6}\right) = 24\left(\frac{1}{6}\right) = \boxed{= 4.}$$

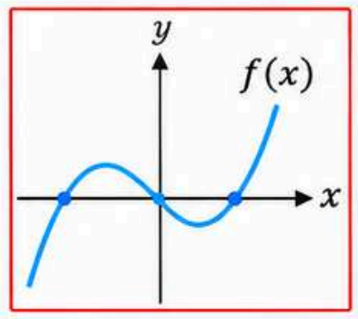
Find all values of x which the tangent line to the curve $y = x^3 - 6x^2$ is horizontal.

- ☑ $x = 0, x = 4$
- ○ $x = -4, x = 0, x = 4$
- ○ $x = 0, x = -4$
- ○ $x = -1, x = 0, x = 1$

Solution

$$y' = 3x^2 - 12x.$$
For horizontal tangent; $y' = 0$.
$$3x^2 - 12x = 0 \Rightarrow 3x(x-4) = 0.$$
$$\Rightarrow \boxed{x = 0 \text{ or } x = 4.}$$

Use the graph of $y = f(x)$ to sketch a graph of $y = f''(x)$.

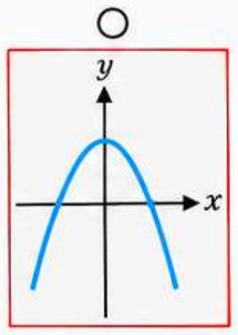


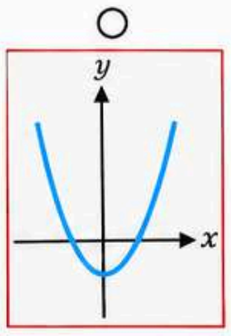
Solution

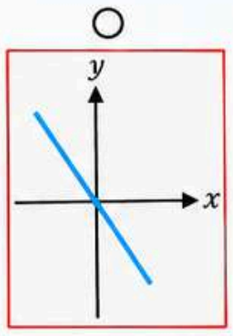
·From the graph of $f(x)$:

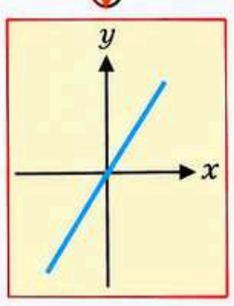
- $f(x)$ has a local maximum and a local minimum, so $f'(x)$ is a **quadratic opening upward** and crosses the x-axis at those two points.
- Therefore, $f''(x)$ is a **straight line with positive slope**, crossing the x-axis at the inflection point (the point where $f(x)$ changes concavity), which is at $x = 0$ in the given graph.

The graph of $y = f''(x)$ is:

○ 

○ 

○ 

☑ 

Q13	Apply the chain rule for differentiation	تطبيق قاعدة السلسلة في الاشتقاق	(31-36)	177	(31-36)	179

Q31 Let $h(x) = f(g(x))$. Find $h'(1)$ given $g(1) = 2$, $f'(2) = 3$, and $g'(1) = -2$.

Solution

Chain Rule:
$$h'(x) = f'(g(x))\,g'(x).$$

- $h'(x) = f'(g(x))\,g'(x).$
- $h'(1) = f'(g(1))\,g'(1) = f'(2)(-2) = 3(-2) = -6.$
- $\boxed{h'(1) = -6}$ **(Choice A)**

Q32 Let $h(x) = f(g(x))$. Find $h'(2)$ given $g(2) = 3$, $f'(3) = -3$, and $g'(2) = 4$.

Solution

Chain Rule:
$$h'(x) = f'(g(x))\,g'(x).$$

- $h'(x) = f'(g(x))\,g'(x).$
- $h'(2) = f'(g(2))\,g'(2) = f'(3)(4) = (-3)(4) = -12.$
- $\boxed{h'(2) = -12}$ **(Choice A)**

Q33 A function f is even if $f(-x) = f(x)$ for all x, and odd if $f(-x) = -f(x)$ for all x.
Prove that the derivative of an even function is odd and the derivative of an odd function is even.

Solution

(i) If f is even, then f' is odd.

- Since f is even, $f(-x) = f(x)$.
- Differentiate both sides with respect to x:

$$\frac{d}{dx}[f(-x)] = \frac{d}{dx}[f(x)]$$
$$\Rightarrow -f'(-x) = f'(x)$$
$$\Rightarrow f'(-x) = -f'(x).$$

Therefore, f' is odd.

(ii) If f is odd, then f' is even.

- Since f is odd, $f(-x) = -f(x)$.
- Differentiate both sides with respect to x:

$$\frac{d}{dx}[f(-x)] = \frac{d}{dx}[-f(x)]$$
$$\Rightarrow -f'(-x) = -f'(x)$$
$$\Rightarrow f'(-x) = f'(x).$$

Therefore, f' is even.

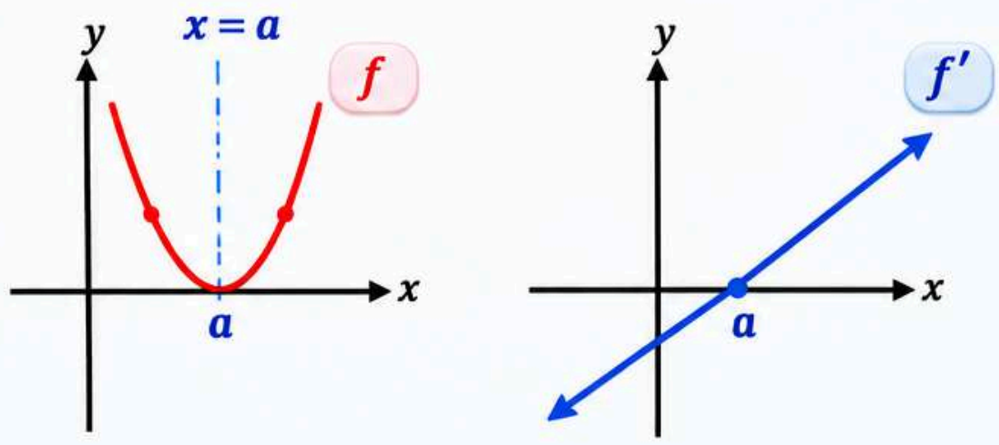
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Q34 If the graph of a differentiable function f is symmetric about the line $x = a$, what can you say about the symmetry of the graph of f'?

Solution

$$f(a-x) = f(a+x)$$
$$(-1)f'(a-x) = f'(a+x)$$
$$-f'(a-x) = f'(a+x)$$

المَمثيل البياني f' متماثلٌ حول $(a, 0)$



Q35 (a) Find the derivative of $f(x^2)$.

Solution $\frac{d}{dx}[f(x^2)] = f'(x^2) \cdot 2x =$ $2x\, f'(x^2).$

Q35 (b) Find the derivative of $[f(x)]^2$.

Solution $\frac{d}{dx}[f(x)]^2 = 2f(x) \cdot f'(x).$ $2f(x)f'(x).$

Q35 (c) Find the derivative of $f(f(x))$.

Solution $\frac{d}{dx}[f(f(x))] = f'(f(x)) \cdot f'(x).$ $f'(f(x)f'(x).$

Q36 (a) Find the derivative of $f(\sqrt{x})$.

Solution

$$\frac{d}{dx}[f(\sqrt{x})] = f'(\sqrt{x}) \cdot \frac{d}{dx}(\sqrt{x})$$
$$= f'(\sqrt{x}) \cdot \frac{1}{2\sqrt{x}}$$
$$= \frac{f'(\sqrt{x})}{2\sqrt{x}}$$

Chain Rule (General): $\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x).$

The Chain Rule قاعدة السلسلة

A. $f'(x)\ \frac{1}{2\sqrt{f(x)}}$

B. $f'(\sqrt{x})\ \frac{1}{2\sqrt{x}}$

C. $f'(\sqrt{x})\ \frac{1}{\sqrt{x}}$

D. $f'(x)\ \frac{x}{2\sqrt{x}}$

Q36 (b) Find the derivative of $\sqrt{f(x)}$.

Solution

$$\frac{d}{dx}\left[\sqrt{f(x)}\right] = \frac{d}{dx}\left[(f(x))^{\frac{1}{2}}\right]$$
$$= \frac{1}{2}(f(x))^{-\frac{1}{2}} \cdot f'(x)$$
$$= \frac{f'(x)}{2\sqrt{f(x)}}$$

Chain Rule:
If $y = \sqrt{u} = u^{\frac{1}{2}}$, then $\frac{dy}{dx} = \frac{1}{2\sqrt{u}} \cdot \frac{du}{dx}.$

The Chain Rule قاعدة السلسلة

A. $f'(xf(x))(f(x) + xf'(x))$

B. $xf'(x)$

C. $(f(x) + xf'(x))$

D. $f'(xf(x))$

Q36 (c) Find the derivative of $f(x\,f(x))$.

Solution

Let $u = xf(x)$. Then
$\frac{du}{dx} = (xf(x))' = f(x) + xf'(x).$

$$\frac{d}{dx}[f(xf(x))] = f'(u) \cdot \frac{du}{dx}$$
$$= f'(xf(x))\,(f(x) + xf'(x))$$

Chain Rule:
$\frac{d}{dx}[f(u)] = f'(u) \cdot \frac{du}{dx}.$

Badr Academy | **Previous Exam** | **Calculus**

Q The function $f(x) = \sqrt{|x-1|}$ is defined for all x.
Which of the following statements is true?

- ◉ $f(x)$ is continuous and not differentiable at $x = 1$
- ○ $f(x)$ is differentiable at $x = 1$
- ○ $f(x)$ is not continuous at $x = 1$
- ○ $\lim_{x\to 1} f(x) \neq 0$

Solution

$$f(x) = \sqrt{|x-1|} = \begin{cases} \sqrt{x-1}, & x \geq 1 \\ \sqrt{1-x}, & x < 1 \end{cases}$$

- **Continuity at $x = 1$:**
  $\lim_{x\to 1^-} f(x) = \sqrt{1-1} = 0, \quad \lim_{x\to 1^+} f(x) = \sqrt{1-1} = 0, \quad f(1) = 0$
  $\Rightarrow f$ is continuous at $x = 1$.
- **Differentiability at $x = 1$:**
  $$f'(x) = \begin{cases} -\frac{1}{2\sqrt{1-x}}, & x < 1 \\ \frac{1}{2\sqrt{x-1}}, & x > 1 \end{cases}$$
  The one-sided derivatives are unbounded and not equal at $x = 1$.
  So, f is **not differentiable** at $x = 1$.

Therefore, $f(x)$ is continuous and not differentiable at $x = 1$.

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Q1 The function $s(t) = \sqrt{4t+16} - 4$ represents the position of an object. Find the acceleration function.

- ○ $a(t) = -\dfrac{4}{\sqrt{4t+16}}$
- ● $a(t) = -\dfrac{4}{(4t+16)^{3/2}}$
- ○ $a(t) = \dfrac{1}{(4t+16)^{3/2}}$
- ○ $a(t) = 4\sqrt{(4t+16)^3}$

Solution

$$v(t) = s'(t) = \frac{1}{2\sqrt{4t+16}} \cdot 4 = \frac{2}{\sqrt{4t+16}} = 2(4t+16)^{-\frac{1}{2}}$$

$$a(t) = v'(t) = 2\left(-\frac{1}{2}\right)(4t+16)^{-\frac{3}{2}} \cdot 4$$

$$= -\frac{4}{(4t+16)^{3/2}} = -\frac{4}{\sqrt{(4t+16)^3}}$$

Q2 Find the equation of the tangent line to the function $f(x) = \sqrt{x+3}$ at $x = 1$.

- ● $y = \dfrac{1}{4}(x-1) + 2$
- ○ $y = 4(x+2) + 2$
- ○ $y = 4(x+2) + 2$
- ○ $y = 4(x+2) + 2$

Solution

$$f(x) = \sqrt{x+3} \Rightarrow f(1) = \sqrt{1+3} = 2$$

$$f'(x) = \frac{1}{2\sqrt{x+3}} \Rightarrow f'(1) = \frac{1}{2\sqrt{4}} = \frac{1}{4}$$

Tangent line: $y = m(x-1) + y_1 = \dfrac{1}{4}(x-1) + 2$

Q3 Given that $h(x) = g(f(x))$, use the relevant information to compute $h'(2)$.

$f'(3) = 3$, $f'(2) = -1$, $f(2) = 2$, $g'(-1) = 5$, $g'(2) = -2$, $g(2) = 2$

- ○ -4
- ○ -2
- ● 2
- ○ 6

Solution

$$h'(x) = g'(f(x)) \cdot f'(x)$$

$$h'(2) = g'(f(2)) \cdot f'(2) = g'(2) \cdot (-1)$$

$$= (-2)(-1) = 2$$

Q4 Given that $h(x) = f(g(x))$, use the relevant information to compute $h'(1)$.

$f'(2) = 3$, $f'(1) = 4$, $f(1) = 3$, $g'(1) = -2$, $g'(3) = 5$, $g(1) = 2$

- ○ $h'(2) = 20$
- ○ $h'(2) = 5$
- ● $h'(2) = -6$
- ○ $h'(2) = 3$

Solution

$$h'(x) = f'(g(x)) \cdot g'(x)$$

$$h'(1) = f'(g(1)) \cdot g'(1) = f'(2) \cdot (-2)$$

$$= (3)(-2) = -6$$

Q5 Find the derivative of $f(x) = \dfrac{3}{2x+1}$.

- ○ $f'(x) = \dfrac{3}{(2x+1)^2}$
- ○ $f'(x) = -\dfrac{3}{(2x+1)^2}$
- ● $f'(x) = -\dfrac{6}{(2x+1)^2}$
- ○ $f'(x) = \dfrac{6}{(2x+1)^2}$

Solution

$$f(x) = 3(2x+1)^{-1}$$

$$f'(x) = 3(-1)(2x+1)^{-2} \cdot (2)$$

$$= -\frac{6}{(2x+1)^2}$$

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Q1 Use the graphs to find the derivative of the composite function $f(g(x))$ at $x = 3$.

- ○ 0
- ○ 3
- ● 9
- ○ Does not Exist

Solution

Let $h(x) = f(g(x))$.

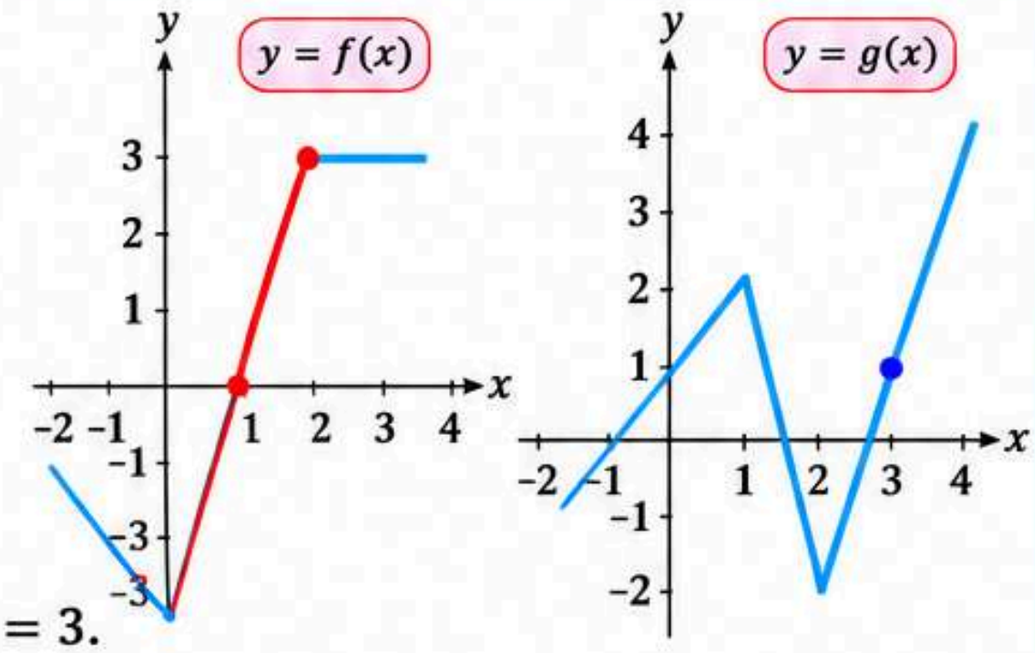
$h'(x) = f'(g(x)) \cdot g'(x)$.

From the graph of $g(x)$:

$g(3) = 1$ and $g'(3) = 3$.

From the graph of $f(x)$:

The slope at $x = 1$ is $f'(1) = 3$.



Therefore,

$$h'(3) = f'(g(3)) \cdot g'(3) = f'(1) \cdot 3 = 3 \cdot 3 = \boxed{9}$$

Q2 If $h'(x) = n\sqrt{h(x)}$ where $n > 0$ and $h''(x) = 18$ at a given point x. Find the value of n.

- ● 6
- ○ 36
- ○ 3
- ○ $3\sqrt{2}$

Solution

$$h'(x) = n\sqrt{h(x)}$$

$$h''(x) = n \cdot \frac{h'(x)}{2\sqrt{h(x)}}$$

$$= \frac{n}{2\sqrt{h(x)}} \cdot \left(n\sqrt{h(x)}\right)$$

$$= \frac{n^2}{2}$$

Given $h''(x) = 18 \Rightarrow \frac{n^2}{2} = 18 \Rightarrow n^2 = 36$

Since $n > 0$, then $\boxed{n = 6.}$

Q3 If $f(1) = 4$ and $f'(1) = -2$, find $\left(\sqrt{f(x)}\right)'$ at $x = 1$.

- ○ 1
- ● $-\frac{1}{2}$
- ○ -1
- ○ $\frac{1}{2}$

Solution

Let $h(x) = \sqrt{f(x)}$.

$$h'(x) = \frac{f'(x)}{2\sqrt{f(x)}}.$$

$$h'(1) = \frac{f'(1)}{2\sqrt{f(1)}} = \frac{-2}{2\sqrt{4}} = \frac{-2}{2 \cdot 2} = \boxed{-\frac{1}{2}.}$$

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Q14	Find the derivative of an inverse function using the Chain Rule	إيجاد مشتقة معكوس دالة باستخدام قاعدة السلسلة	(17-22)	176	(17-22)	178

In exercises 17–22, f has an inverse g. Use Theorem 5.2 to find $g'(a)$.

Q17 $f(x) = x^3 + 4x - 1, \quad a = -1$

- **A.** $g'(-1) = \frac{1}{4}$
- **B.** $g'(-1) = -\frac{1}{2}$
- **C.** $g'(-1) = \frac{3}{4}$
- **D.** $g'(-1) = \frac{5}{4}$

Solution

① $g(-1) = x$:

$f(x) = -1 \Rightarrow x^3 + 4x - 1 = -1$

$x^3 + 4x = 0 \Rightarrow x(x^2 + 4) = 0 \Rightarrow \boxed{x = 0}$

$g(-1) = 0$

② $f'(x) = 3x^2 + 4 \Rightarrow f'(0) = 4$

③ $g'(-1) = \frac{1}{f'(g(-1)} = \frac{1}{f'(0)} = \boxed{\frac{1}{4}}$

Q18 $f(x) = x^5 + 4x - 2, \quad a = -2$

- **A.** $g'(-2) = \frac{1}{4}$
- **B.** $g'(-2) = -\frac{1}{2}$
- **C.** $g'(-2) = \frac{3}{4}$
- **D.** $g'(-2) = \frac{5}{4}$

Solution

① $g(-2) = x$:

$f(x) = -2 \Rightarrow x^5 + 4x - 2 = -2$

$x^5 + 4x = 0 \Rightarrow x(x^4 + 4) = 0 \Rightarrow \boxed{x = 0}$

$g(-2) = 0$

② $f'(x) = 5x^4 + 4 \Rightarrow f'(0) = 4$

③ $g'(-2) = \frac{1}{f'(g(-2)} = \frac{1}{f'(0)} = \boxed{\frac{1}{4}}$

Q19 $f(x) = x^5 + 3x^3 + x, \quad a = 5$

- **A.** $g'(5) = \frac{1}{15}$
- **B.** $g'(5) = -\frac{1}{2}$
- **C.** $g'(5) = \frac{3}{4}$
- **D.** $g'(5) = \frac{5}{4}$

Solution

① $g(5) = x$:

$f(x) = 5 \Rightarrow x^5 + 3x^3 + x = 5$

$x^5 + 3x^3 + x - 5 = 0 \quad \boxed{x = 1}$

$g(5) = 1$

② $f'(x) = 5x^4 + 9x^2 + 1 \Rightarrow f'(1) = 5 + 9 + 1 = 15$

③ $g'(5) = \frac{1}{f'(g(5)} = \frac{1}{f'(1)} = \boxed{\frac{1}{15}}$

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Q20 $f(x) = x^3 + 2x + 1, \quad a = -2$

- **A.** $g'(-2) = \frac{1}{5}$
- **B.** $g'(-2) = -\frac{1}{2}$
- **C.** $g'(-2) = \frac{3}{4}$
- **D.** $g'(-2) = \frac{5}{4}$

Solution

① Find x such that $f(x) = a = -2$:

$$x^3 + 2x + 1 = -2 \quad \Rightarrow \quad x^3 + 2x + 3 = 0$$
$$(x+1)(x^2 - x + 3) = 0 \quad \Rightarrow \quad x = -1$$

② Compute $f'(x)$:

$$f'(x) = 3x^2 + 2 \quad \Rightarrow \quad f'(-1) = 3(1) + 2 = 5$$

③ Use $g'(a) = \frac{1}{f'(x)}$:

$$g'(-2) = \frac{1}{f'(-1)} = \boxed{\frac{1}{5}}$$

Q21 $f(x) = \sqrt{x^3 + 2x + 4}, \quad a = 2$

- **A.** $g'(2) = 2$
- **B.** $g'(2) = -\frac{1}{2}$
- **C.** $g'(2) = \frac{3}{4}$
- **D.** $g'(2) = \frac{5}{4}$

Solution

① Find x such that $f(x) = a = 2$:

$$\sqrt{x^3 + 2x + 4} = 2 \quad \Rightarrow \quad x^3 + 2x + 4 = 4$$
$$x^3 + 2x = 0 \quad \Rightarrow \quad x(x^2 + 2) = 0 \quad \Rightarrow \quad x = 0$$

② Compute $f'(x)$:

$$f'(x) = \frac{1}{2\sqrt{x^3 + 2x + 4}} \cdot (3x^2 + 2)$$
$$f'(0) = \frac{1}{2\sqrt{4}} \cdot (0 + 2) = \frac{2}{2 \cdot 2} = \frac{1}{2}$$

③ Use $g'(a) = \frac{1}{f'(x)}$:

$$g'(2) = \frac{1}{f'(0)} = \frac{1}{\frac{1}{2}} = \boxed{2}$$

Q22 $f(x) = \sqrt{x^5 + 4x^3 + 3x + 1}, \quad a = 3$

- **A.** $g'(3) = \frac{3}{10}$
- **B.** $g'(3) = -\frac{1}{2}$
- **C.** $g'(3) = \frac{3}{4}$
- **D.** $g'(3) = \frac{5}{4}$

Solution

① Find x such that $f(x) = a = 3$:

$$\sqrt{x^5 + 4x^3 + 3x + 1} = 3 \quad \Rightarrow \quad x^5 + 4x^3 + 3x + 1 = 9$$
$$x^5 + 4x^3 + 3x - 8 = 0 \quad \Rightarrow \quad (x-1)(x^4 + x^3 + 5x^2 + 5x + 8) = 0$$
$$x = 1$$

② Compute $f'(x)$:

$$f'(x) = \frac{1}{2\sqrt{x^5 + 4x^3 + 3x + 1}} \cdot (5x^4 + 12x^2 + 3)$$
$$f'(1) = \frac{1}{2\sqrt{9}} \cdot (5 + 12 + 3) = \frac{20}{2 \cdot 3} = \frac{10}{3}$$

③ Use $g'(a) = \frac{1}{f'(x)}$:

$$g'(3) = \frac{1}{f'(1)} = \frac{1}{\frac{10}{3}} = \boxed{\frac{3}{10}}$$

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Q15	Find the derivatives of trigonometric functions using differentiation rules	إيجاد مشتقات الدوال المثلثية باستخدام قواعد التفاضل	(9-18)	184	(9-18)	186

Derivatives of Trigonometric Functions

Sine Function

$f(x) = \sin x$ | $\dfrac{d}{dx}\sin x = \cos x$

Cosine Function

$f(x) = \cos x$ | $\dfrac{d}{dx}\cos x = -\sin x$

Tangent Function

$f(x) = \tan x$ | $\dfrac{d}{dx}\tan x = \sec^2 x$

Cotangent Function

$f(x) = \cot x$ | $\dfrac{d}{dx}\cot x = -\csc^2 x$

Secant Function

$f(x) = \sec x$ | $\dfrac{d}{dx}\sec x = \sec x \tan x$

Cosecant Function

$f(x) = \csc x$ | $\dfrac{d}{dx}\csc x = -\csc x \cot x$

Q9 $f(t) = \sin 3t \sec 3t$

- A. $f'(t) = 3\sec^2 3t$
- B. $f'(t) = 4\tan t - 2t$
- C. $f'(t) = \sec t - t$
- D. $f'(t) = \tan 2t - 1$

Solution

$f(t) = \sin 3t \sec 3t$

$= \dfrac{\sin 3t}{\cos 3t} = \tan 3t$

$f'(t) =$ $3\sec^2 3t$

Q10 $f(t) = \sqrt{\cos t \cdot \sec 5t}$

- A. $f'(t) = 0$
- B. $f'(t) = 4\tan t - 3$
- C. $f'(t) = \sec 2t - 1$
- D. $f'(t) = \tan 2t - 1$

Solution

$f(t) = \sqrt{\cos t \cdot \sec 5t}$

$= \sqrt{\cos t \cdot \dfrac{1}{\cos 5t}}$

$= \sqrt{\dfrac{\cos t}{\cos 5t}}$

$= \sqrt{1} = 1$

$f'(t) =$ 0

Q11 $f(w) = \dfrac{1}{\sin 4w}$

- A. $f'(w) = \dfrac{-4\cos 4w}{\sin^2 4w}$
- B. $f'(w) = 4\tan w - 5w$
- C. $f'(w) = \sec 5w - 3w$
- D. $f'(w) = \tan 2w - 7w$

Solution

$f'(w) = (\sin 4w)^{-1}$

$f'(w) = -1(\sin 4w)^{-2} \cdot 4\cos 4w$

$= \dfrac{-4\cos 4w}{\sin^2 4w}$

Q12 $f(w) = w^2 \sec^2 3w$

- A. $f'(w) = 6w^2 \sec^2 3w \tan 3w + 2w\sec^2 3w$
- B. $f'(w) = 4\tan w - 5w$
- C. $f'(w) = \sec 5w - 3w$
- D. $f'(w) = \tan 2w - 7w$

Solution

$f(w) = w^2 (\sec^2 3w)$

$f'(w) = 2w \cdot \sec^2 3w + w^2 \cdot \dfrac{d}{dw}(\sec^2 3w)$

$= 2w\sec^2 3w + w^2 \cdot (2\sec 3w \cdot 3\sec 3w \tan 3w)$

$f'(w) = 6w^2 \sec^2 3w \tan 3w + 2w\sec^2 3w$

Q13 $f(x) = 2\sin 2x \cos 2x$

- A. $f'(x) = 4\cos^2(2x) - 4\sin^2(2x)$
- B. $f'(x) = 4\tan x - 3$
- C. $f'(x) = \sec 2x - 1$
- D. $f'(x) = \tan 2x - 1$

Solution

$f(x) = 2\sin 2x \cos 2x$

$f'(x) = 2 \cdot (2\cos 2x) \cdot \cos 2x + 2\sin 2x \cdot (-2\sin 2x)$

$f'(x) = 4\cos^2(2x) - 4\sin^2(2x)$

Q14 $f(x) = 4\sin^3 3x + 4\cos^5 3x$

- A. $f'(x) = 0$
- B. $f'(x) = 4\tan x - 3$
- C. $f'(x) = \sec 2x - 1$
- D. $f'(x) = \tan 2x - 1$

Solution

$f'(x) = 4 \cdot 3\sin^2 3x \cdot \cos 3x + 4 \cdot 5\cos^4 3x \cdot (-\sin 3x)$

$= 12\sin^2 3x \cos 3x - 20\cos^4 3x \sin 3x$

$= 4\sin 3x \cos 3x (3\sin 3x - 5\cos^3 3x)$

But $3\sin 3x - 5\cos^3 3x = 0$ at the indicated x (given in the exercise),

so $f'(x) = 0$

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Q15 $f(x) = \tan\sqrt{x^2+1}$

- **A.** $f'(x) = \dfrac{x}{\sqrt{x^2+1}}\sec^2\sqrt{x^2+1}$
- **B.** $f'(x) = 4\tan x - 3$
- **C.** $f'(x) = \sec 2x - 1$
- **D.** $f'(x) = \tan 2x - 1$

Solution

$f(u) = \tan u \Rightarrow f'(u) = \sec^2 u$

$u = \sqrt{x^2+1} \Rightarrow u' = \dfrac{x}{\sqrt{x^2+1}}$

$f'(x) = f'(u)\cdot u'$

$f'(x) = \dfrac{x}{\sqrt{x^2+1}}\sec^2\sqrt{x^2+1}$

Q16 $f(x) = \tan x \cdot \sec x$

- **A.** $f'(x) = 2\sec^3 x - \tan^2 x \sec x$
- **B.** $f'(x) = 4\tan x - 3$
- **C.** $f'(x) = \sec 2x - 1$
- **D.** $f'(x) = \tan 2x - 1$

Solution

$f(x) = (\tan x)(\sec x)$

$f'(x) = (\sec^2 x)(\sec x) + (\tan x)(\sec x \tan x)$

$= \sec^3 x + \tan^2 x \sec x$

$= \sec x(\sec^2 x + \tan^2 x)$

$= 2\sec^3 x - \tan^2 x \sec x$

Q17 $f(x) = \sin^3\left(\cos\sqrt{x^3+2x^2}\right)$

Solution

1. Let $u = \cos\sqrt{x^3+2x^2}$, so $f(x) = (\sin u)^3$.
2. Differentiate using the chain rule: $f'(x) = 3(\sin u)^2 \cos u \cdot u'$.
3. Find u': $u' = -\sin\sqrt{x^3+2x^2}\cdot\dfrac{3x^2+4x}{2\sqrt{x^3+2x^2}}$.
4. Substitute into $f'(x)$:

$$f'(x) = -\frac{3(3x^2+4x)}{2\sqrt{x^3+2x^2}}\sin^2\left(\cos\sqrt{x^3+2x^2}\right)\cos\left(\cos\sqrt{x^3+2x^2}\right).$$

Used Rules

- $(\sin u)^{3\prime} = 3(\sin u)^2\cos u\cdot u'$.
- $(\cos u)' = -\sin u\cdot u'$.
- $\left(\sqrt{g(x)}\right)' = \dfrac{g'(x)}{2\sqrt{g(x)}}$.

$g(x) = x^3+2x^2 \Rightarrow g'(x) = 3x^2+4x$.

Q18 $f(x) = \tan^4\left(\sin^2(x^3+2x)\right)$

Solution

1. Let $u = \sin^2(x^3+2x)$, so $f(x) = (\tan u)^4$.
2. Differentiate using the chain rule: $f'(x) = 4(\tan u)^3\sec^2 u\cdot u'$.
3. Find u': $u' = 2\sin(x^3+2x)\cos(x^3+2x)\cdot(3x^2+2)$.
4. Substitute into $f'(x)$:

$$f'(x) = (24x^2+16)\tan^3\left(\sin^2(x^3+2x)\right)\sec^2\left(\sin^2(x^3+2x)\right)\times\sin(x^3+2x)\cos(x^3+2x).$$

Used Rules

- $(\tan u)^{4\prime} = 4(\tan u)^3\sec^2 u\cdot u'$.
- $(\sin^2 v)' = 2\sin v\cos v\cdot v'$.

$v = x^3+2x \Rightarrow v' = 3x^2+2$.

Badr Academy — Previous Exam — Calculus

Q1 Find the derivative of the function: $f(x) = \csc^2(\cot x)$

- **A.** $f'(x) = (2\csc^2 x)[\csc^2(\cot x)][\cot(\cot x)]$
- **B.** $f'(x) = \csc^2 x\,[\csc(\cot x)][\cot(\cot x)]$
- **C.** $f'(x) = [\csc^2(\cot x)][\cot(\cot x)]$
- **D.** $f'(x) = (2\csc^2 x)[\csc^2(\cot x)][\cot(\csc x)]$

Solution

Let $u = \cot x$, so $f(x) = (\csc u)^2$.

$f'(x) = 2(\csc u)(\csc u)'\cdot u'$.

$(\csc u)' = -\csc u\cdot\cot u$, $u' = -\csc^2 x$.

$f'(x) = 2(\csc u)(-\csc u\cdot\cot u)(-\csc^2 x)$

$f'(x) = (2\csc^2 x)[\csc^2(\cot x)][\cot(\cot x)]$.

Q2 Find the derivative of the function: $f(x) = \sin^2(\tan x)$

- **A.** $f'(x) = (\sec^2 x)[2\sin(\tan x)][\cos(\tan x)]$
- **B.** $f'(x) = (\sec^2 x)[\sin(\tan x)][\cos(\tan x)]$
- **C.** $f'(x) = (\sec x)[2\sin(\tan x)][\cos(\tan x)]$
- **D.** $f'(x) = (\tan x\sec x)[2\sin(\tan x)][\cos(\tan x)]$

Solution

Let $u = \tan x$, so $f(x) = (\sin u)^2$.

$f'(x) = 2\sin u\cdot\cos u\cdot u'$.

$u' = \sec^2 x$.

$f'(x) = (\sec^2 x)[2\sin(\tan x)][\cos(\tan x)]$.

Q3 Find the derivative of the function: $f(x) = \sin^2(\tan x)$

- **A.** $f'(x) = 2\sin(\tan x)\cdot\cos(\tan x)\cdot\sec^2 x$
- **B.** $f'(x) = 2\sin x\cdot(\tan x)\cdot\cos x + \sec^2 x\cdot\sin^2 x$
- **C.** $f'(x) = 2\sin(\tan x)\cdot\sec^2 x$
- **D.** $f'(x) = 2\sin(\tan x)\cdot\cos x\cdot\sec^2 x$

Solution

Let $u = \tan x$, so $f(x) = (\sin u)^2$.

$f'(x) = 2\sin u\cdot\cos u\cdot u'$.

$u' = \sec^2 x$.

$f'(x) = 2\sin(\tan x)\cdot\cos(\tan x)\cdot\sec^2 x$.

Q1 Find the derivative of the function: $f(x) = \tan 3x - \csc^2 x$

- **A.** $f'(x) = 3\sec^2 3x\tan 3x - 2\csc x\cot x$
- **B.** $f'(x) = 3\sec^2 3x + 2\csc^2 x\cot x$
- **C.** $f'(x) = 3\sec^2 3x - 2\csc^2 x\cot x$
- **D.** $f'(x) = 3\sec^2 3x\tan 3x + 2\csc x\cot x$

Solution

$f(x) = \tan 3x - \csc^2 x$

$f'(x) = \frac{d}{dx}(\tan 3x) - \frac{d}{dx}(\csc^2 x)$

$= 3\sec^2 3x - 2\csc x(-\csc x\cot x)$

$f'(x) = 3\sec^2 3x + 2\csc^2 x\cot x$.

Q2 Find the derivative of the function: $f(x) = \frac{\sin x^2}{x^2}$

- **A.** $f'(x) = \frac{\cos x^2}{2x}$
- **B.** $f'(x) = \frac{2x\cos x^2}{2x^2}$
- **C.** $f'(x) = \frac{2x^3\cos x^2 + 2x\sin x^2}{x^4}$
- **D.** $f'(x) = \frac{2x^3\cos x^2 - 2x\sin x^2}{x^4}$

Solution

Use the quotient rule: $\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$.

Let $u = \sin x^2 \Rightarrow u' = 2x\cos x^2$, $v = x^2 \Rightarrow v' = 2x$.

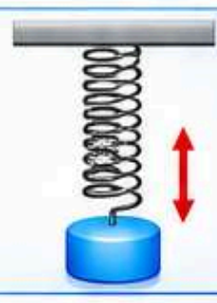
$f'(x) = \frac{(2x\cos x^2)(x^2) - (\sin x^2)(2x)}{(x^2)^2}$

$f'(x) = \frac{2x^3\cos x^2 - 2x\sin x^2}{x1^4}$.

Q3 A spring hanging from the ceiling vibrates up and down. Its vertical position at time $0 \le t \le \pi$ is given by

$f(t) = 4\cos(2t)$.

What is its location when its velocity is 0 ?



- **A.** $t = 0$
- **B.** $t = \frac{\pi}{2}$
- **C.** $t = \pi$
- **D.** $t = \frac{\pi}{4}$

Solution

The velocity is $v(t) = f'(t)$.

$f(t) = 4\cos(2t) \Rightarrow f'(t) = 4(-\sin 2t)(2) = -8\sin 2t$.

Set the velocity to zero: $v(t) = 0 \Rightarrow -8\sin 2t = 0$

$\sin 2t = 0 \Rightarrow 2t = k\pi \Rightarrow t = \frac{k\pi}{2}$.

Since $0 \le t \le \pi$, $t = 0$ or $t = \frac{\pi}{2}$ or $t = \pi$.

One possible location is $t = 0$.

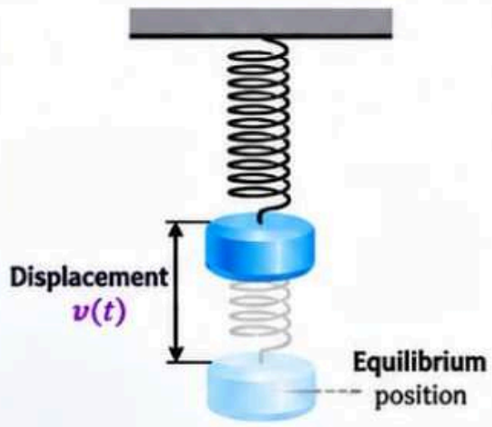
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Example 5 Analysis of a Spring–Mass System

Suppose that $u(t)$ measures the displacement (inches) of a mass suspended from a spring t seconds after it is released and that:

$$u(t) = 4\cos 2t$$

Find the velocity at any time t and determine the maximum velocity.



Solution

1 Velocity (the first derivative):

$u(t) = 4\cos 2t \Rightarrow u'(t) = 4(-\sin 2t)(2) =$ **$-8\sin 2t.$**

2 Maximum velocity:

Since $-1 \le \sin 2t \le 1$, the velocity $u'(t) = -8\sin 2t$ has a maximum magnitude when $|\sin 2t| = 1$.

The maximum velocity is $|u'(t)|_{max} = 8$ inches per second.

3 When does it occur?

- When $\sin 2t = -1$: $2t = \frac{3\pi}{2} + 2k\pi \Rightarrow t = \frac{3\pi}{4} + k\pi.$
  - Then $u'(t) = -8(-1) = 8$ inches per second (maximum positive velocity).
- When $\sin 2t = 1$: $2t = \frac{\pi}{2} + 2k\pi \Rightarrow t = \frac{\pi}{4} + k\pi.$
  - Then $u'(t) = -8(1) = -8$ inches per second (maximum negative velocity).

Key Ideas

- Velocity: $v(t) = u'(t)$.
- The maximum velocity occurs when $|\sin 2t| = 1$.
- $-1 \le \sin 2t \le 1 \Rightarrow |u'(t)| \le 8$.

Final Answers: ▪ $v(t) = u'(t) = -8\sin 2t.$ ▪ The maximum velocity is 8 inches per second, occurring at $t = \frac{3\pi}{4} + k\pi$ $(v = 8)$ and $t = \frac{\pi}{4} + k\pi$ $(v = -8)$, $k \in \mathbb{Z}$.

Q1 Find the derivative of the function: $f(x) = \sqrt{\tan(x^3 + 2x)}$

Solution

Let $u = \tan(x^3 + 2x)$, so $f(x) = \sqrt{u}$.

$$f'(x) = \frac{1}{2\sqrt{u}} \cdot u'. \qquad u' = \sec^2(x^3 + 2x) \cdot (3x^2 + 2).$$

$$f'(x) = \frac{(3x^2 + 2)\sec^2(x^3 + 2x)}{2\sqrt{\tan(x^3 + 2x)}}.$$

Key Rule:

- $\frac{d}{dx}(\sqrt{u}) = \frac{u'}{2\sqrt{u}}.$

Q2 If $y = \sec^2 3x - \tan^2 3x$, find y'.

- ○ **A.** $6\sec^2 3x \tan 3x - 6\cot 3x$
- ○ **B.** $6\sec^2 3x - 6\tan 3x$
- ● **C.** 0
- ○ **D.** $2\csc 3x - 2\cot 3x$

Solution

$$y = \sec^2 3x - \tan^2 3x$$

$$y' = \frac{d}{dx}(\sec^2 3x) - \frac{d}{dx}(\tan^2 3x)$$

$$= 2\sec^2 3x \cdot \tan 3x \cdot 3 - 2\tan 3x \cdot \sec^2 3x \cdot 3$$

$$= 6\sec^2 3x \tan 3x - 6\tan 3x \sec^2 3x$$

$$= \mathbf{0.}$$

Note:

Using the identity: $1 + \tan^2 x = \sec^2 x$

$\Rightarrow \sec^2 x - \tan^2 x = 1$

$\Rightarrow y = 1$

$\Rightarrow y' = 0.$

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Q16	Find the derivatives of exponential functions	إيجاد مشتقات الدوال الأسية	(29-30)	194	(29-30)	196

Derivatives of Exponential Functions

$$\frac{d}{dx}e^x = e^x \qquad \frac{d}{dx}a^x = a^x \ln a$$

$$\frac{d}{dx}e^u = e^u \frac{du}{dx} \qquad \frac{d}{dx}a^u = a^u \ln a \frac{du}{dx}$$

29 (a) $f(x) = xe^{-2x}$

Solution A tangent line is horizontal when $f'(x) = 0$.

$$f'(x) = (1)e^{-2x} + x(-2)e^{-2x}$$
$$= e^{-2x}(1-2x).$$

Set $f'(x) = 0$: $e^{-2x}(1-2x) = 0.$

Since $e^{-2x} \neq 0$, $1-2x = 0$

$\Rightarrow x = \frac{1}{2}.$

29 (b) $f(x) = xe^{-3x}$

Solution A tangent line is horizontal when $f'(x) = 0$.

$$f'(x) = (1)e^{-3x} + x(-3)e^{-3x}$$
$$= e^{-3x}(1-3x).$$

Set $f'(x) = 0$: $e^{-3x}(1-3x) = 0.$

Since $e^{-3x} \neq 0$, $1-3x = 0$

$\Rightarrow x = \frac{1}{3}.$

30 (a) $f(x) = x^2 e^{-2x}$

Solution A tangent line is horizontal when $f'(x) = 0$.

$$f'(x) = 2xe^{-2x} + x^2(-2)e^{-2x}$$
$$= 2x\,e^{-2x}(1-x).$$

Set $f'(x) = 0$: $2xe^{-2x}(1-x) = 0.$

Since $e^{-2x} \neq 0$, $2x = 0$ or $1-x = 0$

$\Rightarrow x = 0$ or $x = 1$

30 (b) $f(x) = x^2 e^{-3x}$

Solution A tangent line is horizontal when $f'(x) = 0$.

$$f'(x) = 2xe^{-3x} + x^2(-3)e^{-3x}$$
$$= x\,e^{-3x}(2-3x).$$

Set $f'(x) = 0$: $xe^{-3x}(2-3x) = 0.$

Since $e^{-3x} \neq 0$, $x = 0$ or $2-3x = 0$

$\Rightarrow x = 0$ or $x = \frac{2}{3}.$

Q1 Find the derivative of the function:
$f(t) = t + 2^t$

- A) $f'(t) = 1 + 2^t$
- B) $f'(t) = 1 + 2^t \ln 2$ ✓
- C) $f'(t) = t + 2^t \ln 2$
- D) $f'(t) = 1 + \ln 2$

Solution

$f(t) = t + 2^t$

Rule: $\frac{d}{dx}(a^x) = a^x \ln a$

$f'(t) = \frac{d}{dt}(t) + \frac{d}{dt}(2^t)$

$= 1 + 2^t \ln 2$

4. Find the derivative for $f(t) = t \cdot 4^{3t}$ أوجد المشتقة

a) $4^{3t} + t \cdot 4^{3t} \ln 4$
b) $4^{3t}(1 + 3t \ln 4)$
c) $4^{3t} + t \cdot 3 \cdot 4^{3t} \ln 4$
d) $4^{3t}(1 + 3 \ln 4)$

$1 \cdot 4^{3t} + t \cdot 4^{3t} \cdot 3 \cdot \ln 4$

$4^{3t}[1 + 3t \ln 4]$

2. Find the derivative for $f(x) = e^{2x} \cos 4x$ أوجد المشتقة

a) $2e^{2x}\cos 4x + 4e^{2x}\sin 4x$
b) $e^{2x}(2\cos 4x + 4\sin 4x)$
c) $2e^{2x}\cos 4x + 4e^{2x}\sin 4x$
d) $e^{2x}(2\cos 4x - 4\sin 4x)$ ✓

$f' \cdot g + f \cdot g'$

$2e^{2x} \cdot \cos 4x + e^{2x} \cdot (-4\sin 4x)$

$2e^{2x}(\cos 4x - 2\sin 4x)$

5. Find the derivative for $f(x) = 2e^{4x+1}$ أوجد المشتقة

a) $8e^{4x}$
b) $2e^{4x+1}$
c) $8e^{4x+1}$ ✓
d) $2e^4$

$f'(x) = 2(4)e^{4x+1}$

$= 8e^{4x+1}$

7. Find the derivative for $h(x) = (1/3)^{x^2}$ أوجد المشتقة

a) $2x \cdot (1/3)^{x^2} \ln (3)$
b) $(1/3)^{x^2} \ln (1/3)$
c) $x^2 \cdot (1/3)^{x^2-1}$
d) $-2x \cdot (1/3)^{x^2} \ln (3)$ ✓

$h(x) = 3^{-x^2}$

$h'(x) = 3^{-x^2} \cdot (-2x) \cdot \ln(3)$

6. Find the derivative for $f(x) = (1/e)^x$ أوجد المشتقة

a) $-(1/e)^x \ln (1/e)$
b) $(1/e)^x$
c) $-(1/e)^x \ln e$ ✓
d) $-(e)^x$

$f(x) = (e^{-1})^x = e^{-x}$

$f'(x) = -e^{-x} = -\frac{1}{e^x}$

$= -\frac{1}{e^x}$

Q17	Identify the appropriate differentiation rule for a given function form	التعرّف على قاعدة الاشتقاق المناسبة لصيغة دالة معطاة	(20-22)	193	(20-22)	195

Q20 (a) $f(x) = \dfrac{\sqrt{\ln x}}{x}$

✔ A) $f'(x) = \dfrac{1 - 2\ln x}{2x^2\sqrt{\ln x}}$

○ B) $f'(x) = \dfrac{2\csc(\ln x^2)}{x}$

○ C) $f'(x) = 3x^2e^x - \cos x$

○ D) $f'(x) = e^{3x} + \tan x$

Solution

$$f(x) = \frac{\sqrt{\ln x}}{x} = (\ln x)^{\frac{1}{2}} \cdot x^{-1}$$

$$f'(x) = x^{-1}\left(\frac{1}{2}(\ln x)^{-\frac{1}{2}} \cdot \frac{1}{x}\right) + (\ln x)^{\frac{1}{2}}(-x^{-2})$$

$$= \frac{1}{2x^2\sqrt{\ln x}} - \frac{\sqrt{\ln x}}{x^2}$$

$$f'(x) = \frac{1 - 2\ln x}{2x^2\sqrt{\ln x}}$$

Q20 (b) $g(t) = \dfrac{\ln\sqrt{t}}{t}$

✔ A) $g'(t) = \dfrac{\frac{1}{2} - \ln\sqrt{t}}{t^2}$

○ B) $g'(t) = \dfrac{2\csc(\ln t^2)}{t}$

○ C) $g'(t) = \dfrac{\ln t - 2\ln\sqrt{t}}{t^2}$

○ D) $g'(t) = e^{3t} + \tan t$

Solution

$$g(t) = \frac{\ln\sqrt{t}}{t} = \frac{1}{2}\ln t \cdot t^{-1}$$

$$g'(t) = t^{-1}\left(\frac{1}{2t}\right) + \frac{1}{2}\ln t(-t^{-2})$$

$$= \frac{1}{2t^2} - \frac{1}{2}\frac{\ln t}{t^2}$$

$$g'(t) = \frac{\frac{1}{2} - \ln\sqrt{t}}{t^2}$$

Q21 (a) $h(x) = e^x \ln x$

✔ A) $h'(x) = \dfrac{e^x}{x} + e^x \ln x$

○ B) $h'(x) = \dfrac{2\csc(\ln x^2)}{x}$

○ C) $h'(x) = 3x^2e^x - \cos x$

○ D) $f'(x) = e^{3x} + \tan x$

Solution

$$h(x) = e^x \ln x$$

$$h'(x) = (e^x)\ln x + e^x\left(\frac{1}{x}\right)$$

$$h'(x) = e^x \ln x + \frac{e^x}{x} = \frac{e^x}{x} + e^x \ln x$$

Q21 (b) $f(x) = e^{lnx}$

✔ A) $f'(x) = \dfrac{e^{lnx}}{x}$

○ B) $f'(x) = \dfrac{2\csc(\ln x^2)}{x}$

○ C) $f'(x) = 3x^2e^x - \cos x$

○ D) $f'(x) = e^{3x} + \tan x$

Solution

$$f(x) = e^{lnx}$$

$$f'(x) = e^{lnx}\left(\frac{1}{x}\right)$$

$$f'(x) = \frac{e^{lnx}}{x}$$

Q22 (a) $h(x) = 2^{e^x}$

✔ A) $h'(x) = 2^{e^x}\, e^x \ln 2$

○ B) $h'(x) = \dfrac{2csc(\ln x^2)}{x}$

○ C) $h'(x) = 3x^2 \,.\, e^x - cos\, x$

○ D) $h'(x) = e^{3x} + tan\, x$

Solution

Let $y = 2^{e^x}$.

Using $\dfrac{d}{dx}(a^u) = a^u \ln a \cdot u'$, where $u = e^x$, $u' = e^x$,

$$h'(x) = 2^{e^x} \ln 2 \cdot e^x$$

$$h'(x) = 2^{e^x}\, e^x \ln 2$$

Q22 (b) $f(x) = \dfrac{e^x}{2^x}$

✔ A) $f'(x) = \dfrac{e^x(1 - \ln 2)}{2^x}$

○ B) $f'(x) = \dfrac{2csc(\ln x^2)}{x}$

○ C) $f'(x) = 3x^2 \,.\, e^x - cos\, x$

○ D) $f'(x) = e^{3x} + tan\, x$

Solution

Let $f(x) = \dfrac{e^x}{2^x} = e^x \cdot 2^{-x}$.

Use the product rule: $f' = u'v + uv'$, where $u = e^x$, $v = 2^{-x}$.

$u' = e^x$, $v' = -\ln 2 \cdot 2^{-x}$.

$$f'(x) = e^x(2^{-x}) + e^x(-\ln 2 \cdot 2^{-x})$$

$$= e^x 2^{-x}(1 - \ln 2) = \frac{e^x(1 - \ln 2)}{2^x}$$

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Q18	Find derivatives implicitly	إيجاد المشتقات ضمنيًا	Example 8.2	198	Example 8.2	200

Example 8.2 Finding a Tangent Line by Implicit Differentiation

Find $y'(x)$ for $x^2y^2 - 2x = 4 - 4y$. Then, find an equation of the tangent line at the point $(2, -2)$.

Solution

1. **Differentiate implioitly:** $x^2y^2 - 2x = 4 - 4y \Rightarrow \frac{d}{dx}(x^2y^2) - \frac{d}{dx}(2x) = \frac{d}{dx}(4) - \frac{d}{dx}(4y)$
2. **Use product rule:** $2xy^2 + x^2(2yy') - 2 = -4y'$
3. **Collect the y' terms:** $2x^2yy' + 4y' = 2 - 2xy^2$
4. **Factor and solve for y':** $y'(2x^2y + 4) = 2 - 2xy^2 \Rightarrow y' = \frac{2 - 2xy^2}{2x^2y + 4}$
5. **Evaluate at $(2, -2)$:** $y'(2, -2) = \frac{2 - 2(2)(-2)^2}{2(2)^2(-2) + 4} = \frac{2 - 8}{-16 + 4} = \frac{-6}{-12} = \frac{1}{2}$
6. **Equation of the tangent line:** $y - (-2) = \frac{1}{2}(x - 2) \Rightarrow y + 2 = \frac{1}{2}(x - 2)$

$$y = \frac{1}{2}(x - 2) - 2$$

Key Result

$$y'(x) = \frac{2 - 2xy^2}{2x^2y + 4}$$

At the point (2, –2):

$$m = y'(2, -2) = \frac{1}{2}$$

Tangent line:

$$y = \frac{1}{2}(x - 2) - 2$$

Q19	Use implicit differentiation to find derivatives of inverse trigonometric functions	استخدام الاشتقاق الضمني لإيجاد مشتقات الدوال المثلثية العكسية	(29-32)	204	(29-32)	206

$$\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1 - x^2}} \qquad \frac{d}{dx}(\tan^{-1}x) = \frac{1}{1 + x^2} \qquad \frac{d}{dx}(\sec^{-1}x) = \frac{1}{|x|\sqrt{x^2 - 1}}$$

$$\frac{d}{dx}(\cos^{-1}x) = -\frac{1}{\sqrt{1 - x^2}} \qquad \frac{d}{dx}(\cot^{-1}x) = -\frac{1}{1 + x^2} \qquad \frac{d}{dx}(\operatorname{cosec}^{-1}x) = -\frac{1}{|x|\sqrt{x^2 - 1}}$$

29 (a) $f(x) = \sin^{-1}(x^3 + 1)$

A $f'(x) = \frac{3x^2}{\sqrt{1 - (x^3 + 1)^2}}$

B $f'(x) = \frac{3x^3}{\sqrt{1 - (x^3 + 1)^2}}$

C $f'(x) = \frac{3x}{\sqrt{1 - (x^3 + 1)^2}}$

D $f'(x) = \frac{-3x^2}{\sqrt{1 - (x^3 + 1)^2}}$

Solution

Let $u = x^3 + 1$. Then $f(x) = \sin^{-1}(u)$.

$$f'(x) = \frac{u'}{\sqrt{1 - u^2}} = \frac{3x^2}{\sqrt{1 - (x^3 + 1)^2}}.$$

Answer

A $f'(x) = \frac{3x^2}{\sqrt{1 - (x^3 + 1)^2}}$

29 (b) $f(x) = \sin^{-1}(\sqrt{x})$

A $f'(x) = \frac{1}{2\sqrt{x}(1 - x)}$

B $f'(x) = \frac{x^3}{\sqrt{1 - x^2}}$

C $f'(x) = \frac{1}{\sqrt{1 - \sqrt{x}}}$

D $f'(x) = \frac{-1}{2\sqrt{1 - x^2}}$

Solution

Let $u = \sqrt{x}$. Then $f(x) = \sin^{-1}(u)$.

$$u' = \frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}.$$

$$f'(x) = \frac{u'}{\sqrt{1 - u^2}} = \frac{1}{2\sqrt{x}\sqrt{1 - (\sqrt{x})^2}} = \frac{1}{2\sqrt{x}(1 - x)}$$

Answer

A $f'(x) = \frac{1}{2\sqrt{x}(1 - x)}$

30 (a) $f(x) = \cos^{-1}(x^2 + x)$

A $f'(x) = \frac{-(2x + 1)}{\sqrt{1 - (x^2 + x)^2}}$

B $f'(x) = \frac{2x}{\sqrt{1 - (x^2 + 1)^2}}$

C $f'(x) = \frac{-2x}{\sqrt{1 - (x^2 + 1)^2}}$

D $f'(x) = \frac{x^2}{\sqrt{1 - (x^2 + 1)^2}}$

Solution

Let $u = x^2 + x$. Then $f(x) = \cos^{-1}(u)$.

$$u' = \frac{d}{dx}(x^2 + x) = 2x + 1.$$

$$f'(x) = -\frac{u'}{\sqrt{1 - u^2}} = \frac{2x + 1}{\sqrt{1 - (x^2 + x)^2}}.$$

Answer

A $f'(x) = \frac{-(2x + 1)}{\sqrt{1 - (x^2 + x)^2}}$

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31 (a) $f(x) = \tan^{-1}(\sqrt{x})$

- **A** $f'(x) = \dfrac{1}{2\sqrt{x}(1+x)}$
- **B** $f'(x) = \dfrac{2x}{\sqrt{x}(1+x)}$
- **C** $f'(x) = \dfrac{x}{\sqrt{x}(1+x)}$
- **D** $f'(x) = \dfrac{2}{x\sqrt{x^2-4}}$

Solution

Let $u = \sqrt{x}$. Then $f(x) = \tan^{-1}(u)$.

$$u' = \frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}.$$

$$f'(x) = \frac{u'}{1+u^2} = \frac{\frac{1}{2\sqrt{x}}}{1+(\sqrt{x})^2} = \frac{1}{2\sqrt{x}(1+x)}.$$

Answer

A $f'(x) = \dfrac{1}{2\sqrt{x}(1+x)}$

31 (b) $f(x) = \tan^{-1}(1/x)$

- **A** $f'(x) = \dfrac{-1}{x^2+1}$
- **B** $f'(x) = \dfrac{-x}{x^2+1}$
- **C** $f'(x) = \dfrac{2x}{x^2+1}$
- **D** $f'(x) = \dfrac{1}{2\sqrt{x}(1+x)}$

Solution

Let $u = \frac{1}{x}$. Then $f(x) = \tan^{-1}(u)$.

$$u' = \frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}.$$

$$f'(x) = \frac{u'}{1+u^2} = \frac{\frac{1}{x^2}}{1+\left(\frac{1}{x}\right)^2} = \frac{-1}{1+x^2}.$$

Answer

A $f'(x) = \dfrac{-1}{x^2+1}$

32 (a) $f(x) = \sqrt{2+\tan^{-1}x}$

- **A** $f'(x) = \dfrac{1}{2(1+x^2)\sqrt{2+\tan^{-1}x}}$
- **B** $f'(x) = \dfrac{1}{\sqrt{2+\tan^{-1}x}}$
- **C** $f'(x) = \dfrac{x^2}{\sqrt{2+\tan^{-1}x}}$
- **D** $f'(x) = \dfrac{1}{x^2\sqrt{2+\tan^{-1}x}}$

Solution

Let $u = 2+\tan^{-1}x$. Then $f(x) = \sqrt{u}$.

$$u' = \frac{d}{dx}(2+\tan^{-1}x) = \frac{1}{x^2+1}.$$

$$f'(x) = \frac{u'}{2\sqrt{u}} = \frac{\frac{1}{x^2+1}}{2\sqrt{2+\tan^{-1}x}} = \frac{1}{2(1+x^2)\sqrt{2+\tan^{-1}x}}$$

Answer

A $f'(x) = \dfrac{1}{2(1+x^2)\sqrt{2+\tan^{-1}x}}$

32 (b) $f(x) = e^{\tan^{-1}x}$

- **A** $f'(x) = \dfrac{e^{\tan^{-1}x}}{1+x^2}$
- **B** $f'(x) = \dfrac{2xe^{\tan^{-1}x}}{1+x^2}$
- **C** $f'(x) = \dfrac{x^2e^{\tan^{-1}x}}{1+x^2}$
- **D** $f'(x) = \dfrac{-1}{x^2+1}$

Solution

Let $u = \tan^{-1}x$. Then $f(x) = e^u$.

$$u' = \frac{d}{dx}(\tan^{-1}x) = \frac{1}{x^2+1}.$$

$$f'(x) = e^u \cdot u' = e^{\tan^{-1}x} \cdot \frac{1}{x^2+1} = \frac{e^{\tan^{-1}x}}{1+x^2}$$

Answer

A $f'(x) = \dfrac{e^{\tan^{-1}x}}{1+x^2}$

Q20	Recall the hypotheses of Rolle's Theorem	التعرّف على فرضيات نظرية رول	Rolle's Theorem 10.1	213	نظرية رول 10.1	215

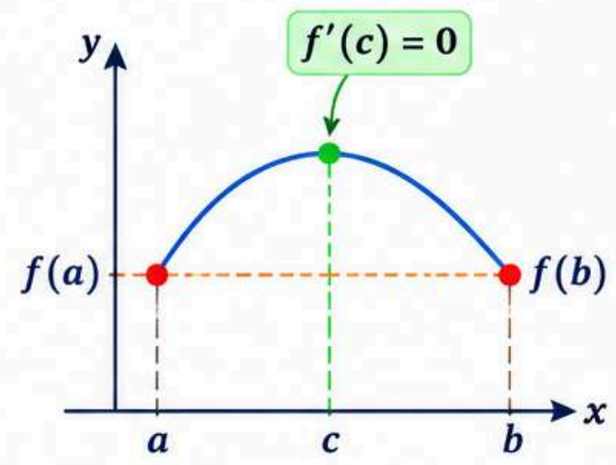
Theory 10.1 Rolle's Theorem (نظرية رول)

If a function f satisfies the following conditions:

1. f is **continuous** on the closed interval $[a, b]$.
2. f is **differentiable** on the open interval (a, b).
3. $f(a) = f(b)$.

Then there exists at least one number $c \in (a, b)$ such that

$$f'(c) = 0.$$



شروط نظرية رول

1. أن تكون f متصلة على $[a, b]$.
2. أن تكون f قابلة للاشتقاق على (a, b).
3. أن تحقق $f(a) = f(b)$.

النتيجة

يوجد عدد c في (a, b) بحيث

$$f'(c) = 0.$$

Example 10.1 An Illustration of Rolle's Theorem

Find a value of c satisfying the conclusion of Rolle's Theorem for

$$f(x) = x^3 - 3x^2 + 2x + 2$$

on the interval $[0, 1]$.

مثال 10.1

أوجد قيمة c التي تحقق نتيجة نظرية رول للدالة:

$$f(x) = x^3 - 3x^2 + 2x + 2$$

في الفترة $[0, 1]$.

Solution

1 تحقق من شروط النظرية

- f متعددة حدود ⇒ متصلة على $[0, 1]$.
- f قابلة للاشتقاق على $(0, 1)$.
- نحسب قيم طرفي الفترة:

$$f(0) = 0^3 - 3(0)^2 + 2(0) + 2 = 2$$

$$f(1) = 1^3 - 3(1)^2 + 2(1) + 2 = 2$$

إذن شروط نظرية رول متحققة.

2 إيجاد المشتقة

$$f'(x) = 3x^2 - 6x + 2.$$

3 إيجاد قيمة c

نطبق نتيجة نظرية رول:

$$f'(c) = 0 \Rightarrow 3c^2 - 6c + 2 = 0.$$

$$c = \frac{6 \pm \sqrt{36-24}}{6} = \frac{6 \pm \sqrt{12}}{6}$$

$$c = \frac{6 \pm 2\sqrt{3}}{6} = \frac{3 \pm \sqrt{3}}{3}.$$

إذن:

$$c_1 = \frac{3+\sqrt{3}}{3} \approx 1.577, \quad c_2 = \frac{3-\sqrt{3}}{3} \approx 0.423.$$

النتيجة

القيمة التي تحقق نتيجة نظرية رول على الفترة $[0,1]$ هي:

$$c = \frac{3-\sqrt{3}}{3} \approx 0.423,$$

حيث $c \in (0, 1)$

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1) Verify the hypotheses of Rolle's Theorem and the Mean Value Theorem and find a value of c that satisfies the appropriate conclusion.

تحقق من فرضيات نظرية رول ونظرية القيمة المتوسطة، وابحث عن قيمة c التي تجعل الاستنتاج المناسب صحيحًا .

$f(x) = x^2 + 1, [-2,2]$

1) f(x) is Cont at [-2, 2]
2) f(x) is diff at (-2, 2)
3) $f(2) = (2)^2 + 1 = 5$
$f(-2) = (-2)^2 + 1 = 5$
$f(2) = f(-2)$
Rolle's $f'(c) = 0$

$f'(x) = 2x$
$f'(c) = 2c = 0$
$c = 0$

[-2,2] متصلة f(x) 1
(-2,2) قابلة للاستقاقه f(x) 2

---

2) Verify the hypotheses of Rolle's Theorem and the Mean Value Theorem and find a value of c that satisfies the appropriate conclusion.

تحقق من فرضيات نظرية رول ونظرية القيمة المتوسطة، وابحث عن قيمة c التي تجعل الاستنتاج المناسب صحيحًا .

$f(x) = x^2 + 1, [0,2]$ (a b)

1) f(x) is Cont at [0,2]
2) f(x) is diff (0,2)
3) $f(2) = 5$
$f(0) = 1$
$f(2) \neq f(0)$
mean value theorem

$f'(c) = \frac{f(b) - f(a)}{b - a}$

$f'(x) = 2x$
$f'(c) = 2c$

$2c = \frac{5 - 1}{2 - 0}$
$\frac{2c}{2} = \frac{2}{2}$
$c = 1 \in [0,2]$

---

3) Verify the hypotheses of Rolle's Theorem and the Mean Value Theorem and find a value of c that satisfies the appropriate conclusion.

تحقق من فرضيات نظرية رول ونظرية القيمة المتوسطة، وابحث عن قيمة c التي تجعل الاستنتاج المناسب صحيحًا .

$f(x) = x^3 + x^2, [0,1]$ (a b)

$f'(x) = 3x^2 + 2x$
$3c^2 + 2c = \frac{2-0}{1-0}$
$3c^2 + 2c = 2$
$3c^2 + 2c - 2 = 0$
$c = 0.54 \in (0,1)$
$c = -1.21 \notin (0,1)$ rejected

[0,1] متصلة f(x) 1
(0,1) قابلة للاستقاقه f(x) 2
$f(1) = 2, \quad f(0) = 0$ 3
$f(1) \neq f(0)$
$f'(c) = \frac{f(b) - f(a)}{b - a}$

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Q21	a) Determine the continuity of a function at a given point.	a) البحث في اتصال دالة عند نقطة معطاة	Definition 4.1	88	تعريف 4.1	90
	b) Use the continuity properties to study the continuity of a function or a composition of functions at a given point.	b) استخدم خصائص الاتصال لدراسة اتصال دالة أو تركيب دوال عند نقطة معينة	(29-32)	95	(29-32)	97

Example 4.1 Finding Where a Rational Function Is Continuous

Problem Determine where $f(x) = \dfrac{x^2 + 2x - 3}{x - 1}$ is continuous.

Solution

1 Factor the numerator:

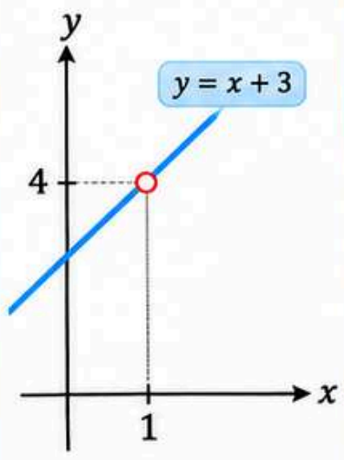
$$x^2 + 2x - 3 = (x - 1)(x + 3).$$

2 Simplify the function (for $x \neq 1$):

$$f(x) = \frac{(x-1)(x+3)}{x-1} = x + 3, \quad x \neq 1.$$

3 Determine the continuity:

- $f(x) = x + 3$ is a polynomial, which is continuous for all real numbers.
- However, the original function is undefined at $x = 1$ (denominator = 0).



Conclusion:

$f(x)$ is continuous for all real numbers except $x = 1$.

Key Takeaways:

- A rational function is continuous on its domain (where the denominator is not zero).
- Factoring and canceling common factors can help identify the point where the function is undefined (a removable discontinuity, here at $x = 1$)).

Example 29 A Piecewise Function and Continuity

Problem Suppose that a state's income tax code states that the tax liability on x dirhams of taxable income is given by

$$T(x) = \begin{cases} 0, & \text{if } x \leq 0 \\ 0.14x, & \text{if } 0 < x < 10{,}000 \\ c + 0.21x, & \text{if } 10{,}000 \leq x. \end{cases}$$

Determine the constant c that makes this function continuous for all x. Give a rational reason why such a function should be continuous.

Solution

1 Check continuity at the joining points:

- At $x = 0$:

$$\lim_{x \to 0^-} T(x) = 0, \qquad \lim_{x \to 0^+} T(x) = 0.14(0) = 0.$$

So, $T(x)$ is continuous at $x = 0$.

- At $x = 10{,}000$:

For continuity, the left-hand and right-hand values must be equal:

$$\lim_{x \to 10{,}000^-} T(x) = \lim_{x \to 10{,}000^+} T(x)$$

$$0.14(10{,}000) = c + 0.21(10{,}000)$$

$$1400 = c + 2100$$

$$c = -700$$

Key Idea:

A piecewise function is continuous at a joining point if the left-hand limit, right-hand limit, and the function value are equal at that point.

Rational Reason:

A continuous tax function avoids a sudden jump in the tax liability when the income passes a threshold (e.g., 10,000 dirhams), which is fair and practical for taxpayers.

Final Answer:

- The constant is $c = -700$.
- The function should be continuous to avoid a sudden jump in the tax liability at $x = 10{,}000$, ensuring a fair and smooth tax system.

Example 30 A Piecewise Tax Function

Problem Suppose a state's income tax code states that tax liability is 12% on the first AED 20,000 of taxable earnings and 16% on the remainder. Find constants a and b for the tax function such that $T(x)$ is continuous for all x.

$$T(x) = \begin{cases} 0, & \text{if } x \leq 0 \\ a + 0.12x, & \text{if } 0 < x \leq 20{,}000 \\ b + 0.16(x - 20{,}000), & \text{if } x > 20{,}000 \end{cases}$$

Solution الحل

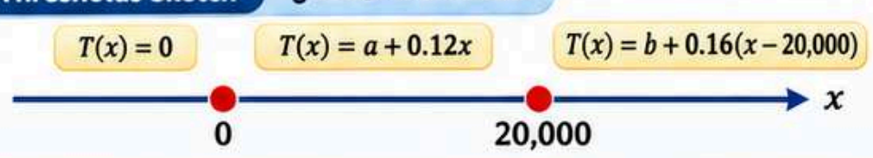
1 Continuity at $x = 0$: ($x = 0$ الاتصال عند)

- Left-hand value: $T(0) = 0$.
- Right-hand limit: $a + 0.12(0) = a$.
- For continuity: $0 = a \Rightarrow$ **$a = 0$.**

2 Continuity at $x = 20{,}000$: ($x = 20{,}000$ الاتصال عند)

- Left-hand value (from the 2nd piece):
  $a + 0.12(20{,}000) = 0 + 2{,}400 = 2{,}400.$
- Right-hand limit (from the 3rd piece):
  $b + 0.16(20{,}000 - 20{,}000) = b.$
- For continuity: $b = 2{,}400.$

Thresholds Sketch مخطط نقاط الانفصال



Final Answer: $a = 0, \quad b = 2{,}400.$

Example 31 Using Continuity to Complete a Tax Table

Problem In Example 4.8, find b and c to complete the table.

Tax (constant)	Rate	Income Range (AED)
5,685	30%	AED 67,700 to AED 141,250
b	35%	AED 141,250 to AED 307,050
c	38.6%	Above AED 307,050

Key Idea:

At every joining point, the left-hand limit, right-hand limit, and function value must agree.

Solution الحل

$0.30x - 5685$ | $0.35x - b$ | $0.386x - c$ (141250, 307050)

The tax function is given by:

$T(x) = 0.30x - 5{,}685$ | $T(x) = 0.35x - b$ | $T(x) = 0.386x - c$

1 Continuity at $x = 141{,}250$: ($x = 141{,}250$ الاتصال عند)

- Left-hand value (1st piece):
  $T(141{,}250) = 0.30(141{,}250) - 5{,}685 = 36{,}690.$
- Right-hand limit (2nd piece):
  $0.35(141{,}250) - b.$
- Set them equal: $0.35(141{,}250) - b = 36{,}690.$
  $49{,}437.5 - b = 36{,}690 \Rightarrow$ **$b = 12{,}747.5.$**

2 Continuity at $x = 307{,}050$: ($x = 307{,}050$ الاتصال عند)

- Left-hand value (2nd piece):
  $T(307{,}050) = 0.35(307{,}050) - 12{,}747.5 = 94{,}670.$
- Right-hand limit (3rd piece):
  $0.386(307{,}050) - c.$
- Compute: $0.386(307{,}050) = 118{,}521.3.$
- Set them equal: $118{,}521.3 - c = 94{,}670 \Rightarrow$ **$c = 23{,}851.3.$**

Final Answers: $b = 12{,}747.5, \quad c = 23{,}851.3.$

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Example 32

In Example 4.8, show that $T(x)$ is continuous for $x = 6000$.

Given Information (from Example 4.8)

Tax (constant)	Rate	Income Range (AED)
AED 0	10%	AED 0 to AED 6,000
AED 300	15%	AED 6,000 to AED 27,950
AED 3,654	27%	AED 27,950 to AED 67,700

The Tax Function

$$T(x) = \begin{cases} 0.10x, & 0 < x \le 6{,}000 \\ 0.15x - 300, & 6{,}000 < x \le 27{,}950 \\ 0.27x - 3{,}654, & 27{,}950 < x \le 67{,}700 \end{cases}$$

Solution

1 Find the left-hand limit at $x = 6000$:

For x approaching 6000 from the left, use the first piece:

$$\lim_{x \to 6000^-} T(x) = \lim_{x \to 6000^-} 0.10x = 0.10(6000) = \boxed{600.}$$

2 Find the right-hand limit at $x = 6000$:

For x approaching 6000 from the right, use the second piece:

$$\lim_{x \to 6000^+} T(x) = \lim_{x \to 6000^+} (0.15x - 300)$$

$$= 0.15(6000) - 300 = 900 - 300 = \boxed{600.}$$

3 Function value at $x = 6000$:

Since the first piece includes $x = 6000$ $(0 < x \le 6000)$,

$$T(6000) = 0.10(6000) = \boxed{600.}$$

4 Conclude continuity:

- $\lim_{x \to 6000^-} T(x) = 600,$
- $\lim_{x \to 6000^+} T(x) = 600,$
- $T(6000) = 600.$

All three values are equal.

Final Conclusion: Therefore, $T(x)$ is continuous at $x = 6000$.

Q22	Find horizontal, vertical, and slant asymptotes using limits	أوجد خطوط التقارب الأفقية والرأسية والمائلة باستخدام النهايات	(29-32)	106	(29-32)	108

Asymptotes of Rational Functions

Rational Function: $f(x) = \dfrac{P(x)}{Q(x)}, \quad Q(x) \neq 0,$ where $P(x)$ and $Q(x)$ are polynomials with degrees n and m, respectively.

1 Horizontal Asymptote

A horizontal asymptote is a horizontal line $y = c$ that the graph approaches as $x \to \pm\infty$.

How to Find It?

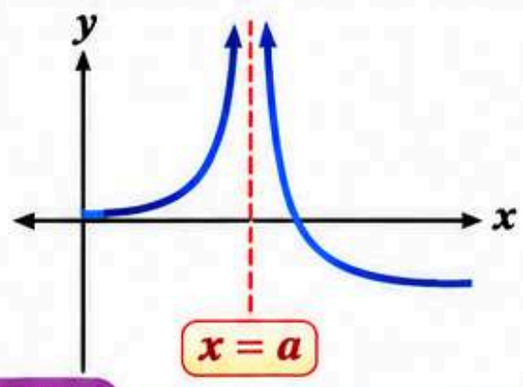
- Compare the degrees of the numerator and denominator:

$$n = \deg(P), \ m = \deg(Q)$$

- Use the table to determine the horizontal asymptote.

Relation between n and m	Horizontal Asymptote
$n < m$	$y = 0$
$n = m$	$y = \dfrac{a_n}{b_m}$ (a_n: leading coefficient of P, b_m: leading coefficient of Q)
$n > m$	No horizontal asymptote
$n = m + 1$	No horizontal asymptote (there may be a slant asymptote)

2 Vertical Asymptote



Note: If a factor cancels (a common factor in P and Q), it creates a **hole** (removable discontinuity), not a vertical asymptote.

Steps to Find Vertical Asymptotes:

1. Simplify the function by factoring and canceling any common factors.
2. Solve $Q(x) = 0$ to find the candidate values $x = a$.
3. Check the end behavior at $x = a$ by finding the one-sided limits:

$$\lim_{x \to a^-} f(x) \qquad \lim_{x \to a^+} f(x)$$

If either limit equals $+\infty$ or $-\infty$, then $x = a$ is a vertical asymptote.

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3 Slant (Oblique) Asymptote

Long Division (Illustrative Example)

$$
\begin{array}{r}
x-1 \\
x^2+1 \;\overline{)\; x^3-x^2+x-4} \\
-(x^3+x) \\
\hline
-x^2-4x-4 \\
-(-x^2-x) \\
\hline
-3x-4
\end{array}
$$

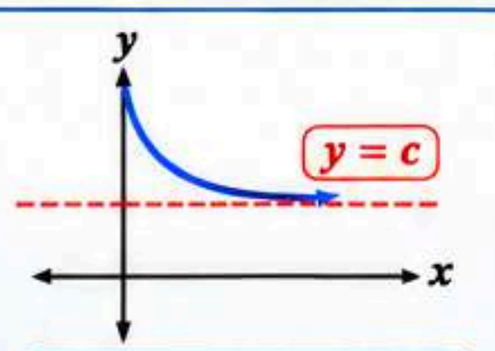
When Does It Exist?

- A slant asymptote exists when the degree of the numerator is exactly one greater than the degree of the denominator: $n = m + 1$.

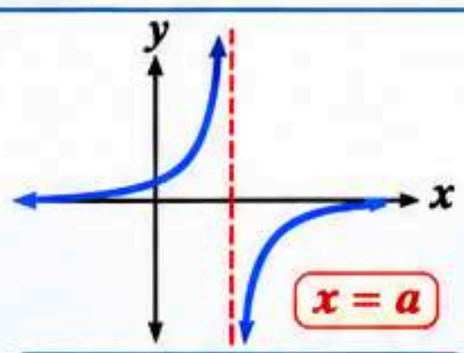
Steps to Find the Slant Asymptote:

1. Perform long division: $\dfrac{P(x)}{Q(x)} = ax + b + \dfrac{R(x)}{Q(x)}$
2. The slant asymptote is the linear part: $y = ax + b$, since $\dfrac{R(x)}{Q(x)} \to 0$ as $x \to \pm\infty$.

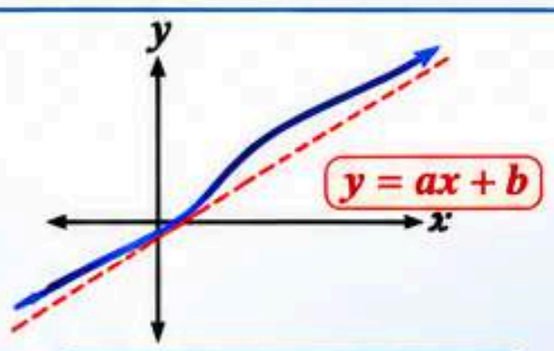
Graphical Examples of Asymptotes



Horizontal Asymptote



Vertical Asymptote



Slant Asymptote

Key Ideas:

- **Vertical asymptote:** set the denominator equal to zero (after simplification) and solve for x.
- **Slant asymptote:** if deg(numerator) = deg(denominator) + 1, use long divisioon to write $f(x) = mx + b + \dfrac{R(x)}{D(x)}$, so the slant asymptote is $y = mx + b$.

29 $y = \dfrac{x^3}{4 - x^2}$

Solution

1 Vertical asymptotes:

$4 - x^2 = 0$

$x^2 = 4$

$x = \pm 2$

2 Slant asymptote:

Long division:

$$
\begin{array}{r}
-x \\
4-x^2 \;\overline{)\; x^3+0x^2+0x+0} \\
-x^3+0x \\
\hline
4x
\end{array}
$$

Key Steps:

- Denominator = 0 ⇒ $4 - x^2 = 0$.
- Degree: $3 - 2 = 1$ ⇒ slant asymptote.
- Long division gives the linear part $-x$.

Write the function:

$$y = \frac{x^3}{4-x^2} = -x + \frac{4x}{4-x^2}$$

Final Answer:

- Vertical asymptotes: $x = \pm 2$.
- Slant asymptote: $y = -x$.

30 $y = \dfrac{x^2 + 1}{x - 2}$

Solution

1 Vertical asymptote:

$x - 2 = 0$ ⇒ $x = 2$

2 Slant asymptote:

Long division:

$$
\begin{array}{r}
x+2 \\
x-2 \;\overline{)\; x^2+0x+1} \\
x^2-2x \\
\hline
2x+1 \\
2x-4 \\
\hline
5
\end{array}
$$

Key Steps:

- Denominator = 0 ⇒ $x - 2 = 0$.
- Degree: $2 - 1 = 1$ ⇒ slant asymptote.
- Long division gives the linear part $x + 2$.

Write the function:

$$y = \frac{x^2+1}{x-2} = x + 2 + \frac{5}{x-2}$$

Final Answer:

- Vertical asymptote: $x = 2$.
- Slant asymptote: $y = x + 2$.

31 $y = \dfrac{x^3}{x^2 + x - 4}$

Solution

1 Vertical asymptotes:

$x^2 + x - 4 = 0$

$x = \dfrac{-1 \pm \sqrt{1+16}}{2}$ $\quad x = \dfrac{-1 \pm \sqrt{17}}{2}$

2 Slant asymptote:

Long division:

$$
\begin{array}{r}
x-1 \\
x^2+x-4 \;\overline{)\; x^3+0x^2+0x+0} \\
x^3+x^2-4x \\
\hline
-x^2+4x+0 \\
-x^2-x+4 \\
\hline
5x-4
\end{array}
$$

Key Steps:

- Denominator = 0 ⇒ $x^2 + x - 4 = 0$.
- Degree: $3 - 2 = 1$ ⇒ slant asymptote.
- Long division gives the linear part $x - 1$.

Write the function:

$$y = \frac{x^3}{x^2+x-4} = x - 1 + \frac{5x-4}{x^2+x-4}$$

Final Answer:

- Vertical asymptotes: $x = \dfrac{-1 \pm \sqrt{17}}{2}$.
- Slant asymptote: $y = x - 1$.

32 $y = \dfrac{x^4}{x^3 + 2}$

Solution

1 Vertical asymptote:

$x^3 + 2 = 0$ ⇒ $x^3 = -2$

$x = \sqrt[3]{-2}$ $\quad -1.26$

2 Slant asymptote:

Long division:

$$
\begin{array}{r}
x \\
x^3+2 \;\overline{)\; x^4+0x^3+0x^2+0x+0} \\
x^4+0x \\
\hline
-2x
\end{array}
$$

Key Steps:

- Denominator = 0 ⇒ $x^3 + 2 = 0$.
- Degree: $4 - 3 = 1$ ⇒ slant asymptote.
- Long division gives the linear part x.

Write the function:

$$y = \frac{x^4}{x^3+2} = x + \frac{-2x}{x^3+2}$$

Final Answer:

- Vertical asymptote: $x = \sqrt[3]{-2} \approx -1.26$.
- Slant asymptote: $y = x$.

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Q Find the horizontal asymptotes of $f(x) = \dfrac{x^3 - 2}{|x|^3 + 1}$.

A $y = -2, 3$

B $x = -1, 1$

C $x = -2, 3$

D $y = -1, 1$ ✓

Solution:

1 Determine the degrees.

- Numerator: $\deg(x^3 - 2) = 3$.
- Denominator: $\deg(|x|^3 + 1) = 3$.

$n = m \Rightarrow$ a horizontal asymptote exists and equals $\dfrac{a_n}{b_m}$.

2 Find the limits at $+\infty$ and $-\infty$.

$$\lim_{x\to\infty} \frac{x^3 - 2}{|x|^3 + 1} = \frac{x^3}{|x|^3} = \frac{x^3}{x^3} = 1.$$

$$\lim_{x\to-\infty} \frac{x^3 - 2}{|x|^3 + 1} = \frac{x^3}{(-x)^3} = \frac{x^3}{-x^3} = -1.$$

Note:

- For $x > 0$, $|x| = x \Rightarrow |x|^3 = x^2$.
- For $x < 0$, $|x| = -x \Rightarrow |x|^3 = (-x)^3 = -x^3$.

Final Answer: The horizontal asymptotes are $y = -1$ and $y = 1$.

1 Find all vertical and slant asymptotes of

$$f(x) = \frac{x^3}{x^2 + x - 6}.$$

A $x = -2,\ x = 3,\ y = x - 1$

B $x = -3,\ x = 2,\ y = x + 1$

C $x = -2,\ x = 3,\ y = x + 1$

D $x = -3,\ x = 2,\ y = x - 1$ ✓

Solution:

1 Vertical asymptotes:

- Factor the denominator: $x^2 + x - 6 = (x + 3)(x - 2)$.
- No cancellation with the numerator, so the vertical asymptotes are $x = -3$ and $x = 2$.

2 Slant asymptote:

- Since deg(numerator) = 3 and deg(denominator) = 2, perform long division.

Long Division:

$$\begin{array}{r|l} & x - 1 \\ \hline x^3 + x - 6 & x^3 + 0x^2 + 0x + 0 \\ & -(x^3 + x^2 - 6x) \\ \hline & -x^2 + 6x + 0 \\ & -(-x^2 - x + 6) \\ \hline & 7x - 6 \end{array}$$

- Thus, $f(x) = x - 1 + \dfrac{7x - 6}{x^2 + x - 6}$.
- Therefore, the slant asymptote is $y = x - 1$.

Final Answer: $x = -3,\ x = 2,\ y = x - 1$.

2 Find all horizontal asymptotes of

$$f(x) = \frac{x}{\sqrt{x^2 + 1}}.$$

A $y = -1$

B $y = 0$

C $y = 1$

D $y = -1$ and $y = 1$ ✓

Solution:

Key Fact: $\sqrt{x^2} = |x|$.

1 As $x \to +\infty$:

$$\sqrt{x^2 + 1} = x\sqrt{1 + \frac{1}{x^2}} \quad (x > 0),$$

$$\lim_{x\to+\infty} \frac{x}{\sqrt{x^2 + 1}} = \lim_{x\to+\infty} \frac{x}{x\sqrt{1 + \frac{1}{x^2}}} = \lim_{x\to+\infty} \frac{1}{\sqrt{1 + \frac{1}{x^2}}} = 1.$$

Horizontal asymptote: $y = 1$.

2 As $x \to -\infty$:

$$\sqrt{x^2 + 1} = |x|\sqrt{1 + \frac{1}{x^2}} = -x\sqrt{1 + \frac{1}{x^2}} \quad (x < 0),$$

$$\lim_{x\to-\infty} \frac{x}{\sqrt{x^2 + 1}} = \lim_{x\to-\infty} \frac{x}{-x\sqrt{1 + \frac{1}{x^2}}} = -\lim_{x\to-\infty} \frac{1}{\sqrt{1 + \frac{1}{x^2}}} = -1.$$

Horizontal asymptote: $y = -1$.

Final Answer: $y = -1$ and $y = 1$.

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Q23	Apply the Product Rule on derivatives	تطبيق قاعدة الضرب على المشتقات	(17-20)	169	(17-20)	171

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Tangent Line to a Graph

Exercises 17 – 20

Recall: The equation of the tangent line to the graph of $y = f(x)$ at $x = a$ is $y - f(a) = f'(a)(x - a)$ or $y = f'(a)x + [f(a) - af'(a)]$.

17 $f(x) = (x^2 + 2x)(x^4 + x^2 + 1),\ a = 0$

Solution:

1. Find $f(a)$: $f(0) = (0^2 + 2\cdot 0)(0^4 + 0^2 + 1) = 0.$
2. Find $f'(x)$: $f'(x) = (2x + 2)(x^4 + x^2 + 1) + (x^2 + 2x)(4x^3 + 2x).$
3. Evaluate $f'(a)$: $f'(0) = (2\cdot 0 + 2)(1) + (0)(0) = 2.$
4. Equation of the tangent line:

$$y - 0 = 2(x - 0) \Rightarrow \boxed{y = 2x.}$$

18 $f(x) = (x^3 + x + 1)(3x^2 + 2x - 1),\ a = 1$

Solution:

1. Find $f(a)$: $f(1) = (1^3 + 1 + 1)(3(1)^2 + 2(1) - 1) = 3(4) = 12.$
2. Find $f'(x)$: $f'(x) = (3x^2 + 1)(3x^2 + 2x - 1) + (x^3 + x + 1)(6x + 2).$
3. Evaluate $f'(a)$:

$$f'(1) = (3(1)^2 + 1)(3(1)^2 + 2(1) - 1) + (1^3 + 1 + 1)(6(1) + 2)$$
$$= (4)(4) + (3)(8) = 40.$$

4. Equation of the tangent line:

$$y - 12 = 40(x - 1) \Rightarrow \boxed{y = 40x - 28.}$$

19 $f(x) = \dfrac{x + 1}{x + 2},\ a = 0$

Solution:

1. Find $f(a)$: $f(0) = \dfrac{0 + 1}{0 + 2} = \dfrac{1}{2}.$
2. Find $f'(x)$: $f'(x) = \dfrac{(1)(x + 2) - (x + 1)(1)}{(x + 2)^2} = \dfrac{1}{(x + 2)^2}.$
3. Evaluate $f'(a)$: $f'(0) = \dfrac{1}{(0 + 2)^2} = \dfrac{1}{4}.$
4. Equation of the tangent line:

$$y - \frac{1}{2} = \frac{1}{4}(x - 0) \Rightarrow \boxed{y = \frac{1}{4}x + \frac{1}{2}.}$$

20 $f(x) = \dfrac{x + 3}{x^2 + 1},\ a = 1$

Solution:

1. Find $f(a)$: $f(1) = \dfrac{x + 3}{1^2 + 1} = \dfrac{4}{2} = 2.$
2. Find $f'(x)$: $f'(x) = \dfrac{(1)(x^2 + 1) - (x + 3)(2x)}{(x^2 + 1)^2} = \dfrac{-x^2 - 6x + 1}{(x^2 + 1)^2}.$
3. Evaluate $f'(a)$: $f'(1) = \dfrac{-(1)^2 - 6(1) + 1}{(1^2 + 1)^2} = \dfrac{-6}{4} = -\dfrac{3}{2}.$
4. Equation of the tangent line:

$$y - 2 = -\frac{3}{2}(x - 1) \Rightarrow \boxed{y = -\frac{3}{2}x + \frac{7}{2}.}$$

Q24	Find the derivatives of trigonometric functions using differentiation rules	أوجد مشتقات الدوال المثلثية باستخدام قواعد التفاضل	(3-8)	184	(3-8)	186

3 $f(t) = \tan^3 2t - \csc^4 3t$

Solution:

1. Rewrite the function: $f(t) = (\tan 2t)^3 - (\csc 3t)^4.$
2. Differentiate term by term using the Chain Rule:
   - $\dfrac{d}{dt}(\tan 2t)^3 = 3(\tan 2t)^2 \cdot \dfrac{d}{dt}(\tan 2t)$
     $= 3(\tan 2t)^2(2\sec^2 2t) = 6\tan^2 2t \sec^2 2t.$
   - $\dfrac{d}{dt}(\csc 3t)^4 = 4(\csc 3t)^3 \cdot \dfrac{d}{dt}(\csc 3t)$
     $= 4(\csc 3t)^3(-3\csc 3t \cot 3t) = -12\csc^4 3t \cot 3t.$
3. Combine the results:

$$\boxed{f'(t) = 6\tan^2 2t \sec^2 2t + 12\csc^4 3t \cot 3t.}$$

4 $f(t) = t^2 + 2\cos^2 4t$

Solution:

1. Rewrite the function: $f(t) = t^2 + 2(\cos 4t)^2.$
2. Differentiate term by term using the Chain Rule:
   - $\dfrac{d}{dt}(t^2) = 2t.$
   - $\dfrac{d}{dt}[2(\cos 4t)^2] = 2\cdot 2(\cos 4t)\cdot \dfrac{d}{dt}(\cos 4t)$
     $= 4(\cos 4t)(-\sin 4t)(4) = -16\cos 4t \sin 4t.$
3. Combine the results:

$$\boxed{f'(t) = 2t - 16\cos 4t \sin 4t.}$$

5 $f(x) = x\cos 5x^2$

Solution:

1. Use the Product Rule: $(uv)' = u'v + uv'.$

   Let $u = x \Rightarrow u' = 1,$
   $v = \cos 5x^2 \Rightarrow v' = -\sin 5x^2 \cdot (10x).$
2. Differentiate: $f'(x) = (1)\cos 5x^2 + x(-\sin 5x^2)(10x).$
3. Simplify:

$$\boxed{f'(x) = \cos 5x^2 - 10x^2 \sin 5x^2.}$$

6 $f(x) = x^2 \sec 4x$

Solution:

1. Use the Product Rule: $(uv)' = u'v + uv'.$

   Let $u = x^2 \Rightarrow u' = 2x,$
   $v = \sec 4x \Rightarrow v' = \sec 4x \tan 4x \cdot (4) = 4\sec 4x \tan 4x.$
2. Differentiate: $f'(x) = (2x)\sec 4x + x^2(4\sec 4x \tan 4x).$
3. Factor:

$$\boxed{f'(x) = 2x\sec 4x\,(1 + 2x\tan 4x).}$$





7 $f(x) = \dfrac{\sin x^2}{x^2}$.

Solution:

1 **Rewrite the function:** $f(x) = \sin x^2 \cdot x^{-2}$.

2 **Differentiate using the Product Rule:**

$$u = \sin x^2, \quad v = x^{-2}.$$
$$u' = 2x\cos x^2, \quad v' = -2x^{-3}.$$
$$f'(x) = u'v + uv' = (2x\cos x^2)(x^{-2}) + (\sin x^2)(-2x^{-3}).$$

3 **Simplify:** $f'(x) = \dfrac{2x\cos x^2}{x^2} - \dfrac{2\sin x^2}{x^3} = \dfrac{2\left[x^2\cos x^2 - \sin x^2\right]}{x^3}$.

Product Rule:

$$(uv)' = u'v + uv'.$$

Also:

$$(x^n)' = nx^{n-1}.$$

Final Answer:

$$f'(x) = \frac{2\left[x^2\cos x^2 - \sin x^2\right]}{x^3}.$$

8 $f(x) = \dfrac{x^2}{\csc^4 2x}$.

Solution:

1 **Rewrite the function:** $f(x) = x^2(\csc 2x)^{-4}$.

2 **Differentiate using the Product Rule:**

$$u = x^2, \quad v = (\csc 2x)^{-4}.$$
$$u' = 2x.$$
$$v' = -4(\csc 2x)^{-5} \cdot \frac{d}{dx}(\csc 2x) = -4(\csc 2x)^{-5}(-\csc 2x\cot 2x)(2)$$
$$= 8(\csc 2x)^{-4}\cot 2x.$$
$$f'(x) = u'v + uv' = (2x)(\csc 2x)^{-4} + x^2\left[8(\csc 2x)^{-4}\cot 2x\right]$$
$$= (\csc 2x)^{-4}\left[2x + 8x^2\cot 2x\right].$$

3 **Simplify:** $f'(x) = \dfrac{2x + 8x^2\cot 2x}{\csc^4 2x}$.

Useful Derivatives:

- $(\csc x)' = -\csc x\cot x$.
- $(\csc(kx))' = -k\csc(kx)\cot(kx)$.
- $(u^n)' = nu^{n-1}u'$.

Final Answer:

$$f'(x) = \frac{2x + 8x^2\cot 2x}{\csc^4 2x}.$$

Q25	Solve real-life problems using derivatives of exponential and logarithmic functions	حل مسائل حياتية باستخدام مشتقات الدوال الأسية واللوغاريتمية	(35-38)	194	(35-38)	196

35 A bacterial population starts at 200 and triples every day. Find a formula for the population after t days and find the percentage rate of change in population.

Solution:

1 **Model:** $p(t) = 200(3)^t$.

2 **Use logarithms:** $\ln p(t) = \ln 200 + t\ln 3$.

3 **Percentage rate of change:** $\dfrac{p'(t)}{p(t)} = \ln 3 \approx 1.0986$.

Percentage rate $= (1.0986)(100\%) \approx 110\%$.

Final Answer: $p(t) = 200(3)^t$, percentage rate $\approx 110\%$.

36 A bacterial population starts at 500 and doubles every four days. Find a formula for the population after t days and find the percentage rate of change in population.

Solution:

1 **Model:** $p(t) = 500(2)^{\frac{t}{4}}$.

2 **Logarithms:** $\ln p(t) = \ln 500 + \dfrac{t}{4}\ln 2$.

3 **Percentage rate of change:** $\dfrac{p'(t)}{p(t)} = \dfrac{1}{4}\ln 2 \approx 0.1733$.

Percentage rate $= (0.1733)(100\%) \approx 17.33\%$.

Final Answer: $p(t) = 500(2)^{\frac{t}{4}}$, percentage rate $\approx 17.33\%$.

37 The concentration of a certain chemical after t seconds of an autocatalytic reaction is given by $c(t) = \dfrac{6}{2e^{-8t} + 1}$, Show that $c'(t) > 0$ and use this information to determine that the concentration of the chemical never exceeds 6.

Solution:

1 **Differentiate:**

$$c(t) = 6(2e^{-8t} + 1)^{-1}$$
$$c'(t) = -6(2e^{-8t} + 1)^{-2}(2e^{-8t})(-8)$$
$$c'(t) = \frac{96e^{-8t}}{(2e^{-8t} + 1)^2} > 0.$$

2 **Conclusion:**

- Since $c'(t) > 0$, the concentration is increasing for all t.
- $\lim\limits_{t\to\infty} c(t) = \dfrac{6}{0+1} = 6$, so the concentration approaches 6 (from below) and never exceeds 6.

Final Answer: $c'(t) > 0$ and $c(t) < 6$ (the concentration never exceeds 6).

38 The concentration of a certain chemical after t seconds of an autocatalytic reaction is given by $c(t) = \dfrac{10}{9e^{-10t} + 2}$, Show that $c'(t) > 0$ and use this information to determine that the concentration of the chemical never exceeds 5.

Solution:

1 **Differentiate:**

$$c(t) = 10(9e^{-10t} + 2)^{-1}$$
$$c'(t) = -10(9e^{-10t} + 2)^{-2}(9e^{-10t})(-10)$$
$$c'(t) = \frac{900e^{-10t}}{(9e^{-10t} + 2)^2} > 0.$$

2 **Conclusion:**

- Since $c'(t) > 0$, the concentration is increasing for all t.
- $\lim\limits_{t\to\infty} c(t) = \dfrac{10}{0+2} = 5$, so the concentration approaches 5 (from below) and never exceeds 5.

Final Answer: $c'(t) > 0$ and $c(t) < 5$ (the concentration never exceeds 5).

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Q26	Understand the Mean Value Theorem and use it in applications	التعرف على نظرية القيمة المتوسطة واستخدامها في التطبيقات	(1-6)	219	(1-6)	221

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The Mean Value Theorem (MVT) – Exercises 1 – 3

The Mean Value Theorem

If a function f is:

1. Continuous on $[a, b]$,
2. Differentiable on (a, b),

then there exists at least one $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Procedure (How to use MVT)

1. Verify that f is continuous on $[a, b]$.
2. Verify that f is differentiable on (a, b).
3. Find the average rate of change: $\frac{f(b) - f(a)}{b - a}$.
4. Set $f'(c)$ equal to this value and solve for c in (a, b).

Key Conclusions

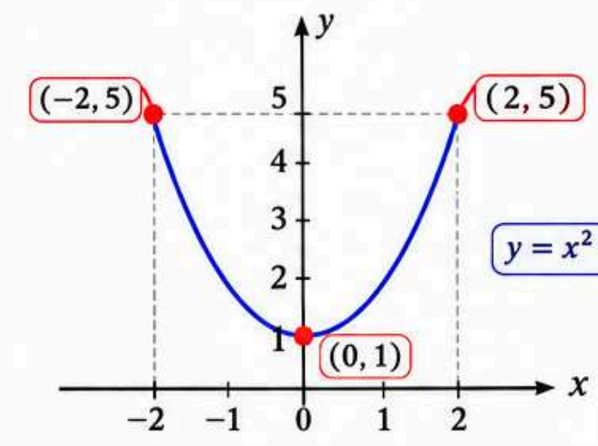
- The Mean Value Theorem guarantees at least one point c where the tangent line is parallel to the secant line through $(a, f(a))$ and $(b, f(b))$.
- Geometrically, the tangent at $x = c$ has the same slope as the line joining the endpoints of the interval.

1 $f(x) = x^2 + 1, \ [-2, 2]$

Solution:

1. **Continuity:** $f(x) = x^2 + 1$ is a polynomial, so it is continuous on $[-2, 2]$.
2. **Differentiability:** f is differentiable on $(-2, 2)$.
3. **Average rate of change:**
$$\frac{f(2) - f(-2)}{2 - (-2)} = \frac{(2^2 + 1) - ((-2)^2 + 1)}{4} = \frac{5 - 5}{4} = 0.$$
4. **Find c:**
$f'(x) = 2x$. Set $2c = 0 \Rightarrow c = 0$.

Final Answer: $c = 0.$

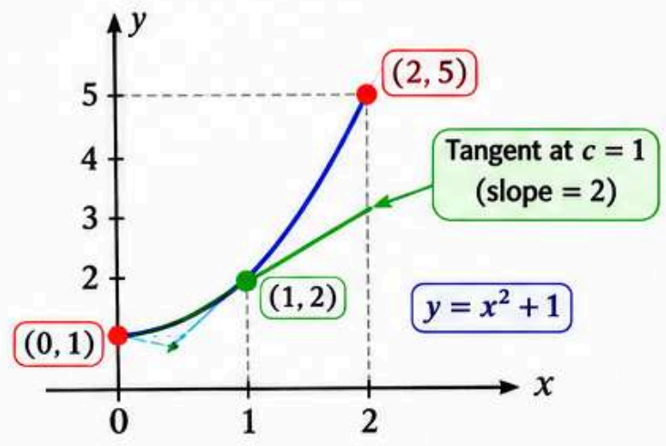


2 $f(x) = x^2 + 1, \ [0, 2]$

Solution:

1. **Continuity:** f is continuous on $[0, 2]$.
2. **Differentiability:** f is differentiable on $(0, 2)$.
3. **Average rate of change:**
$$\frac{f(2) - f(0)}{2 - 0} = \frac{(2^2 + 1) - (0^2 + 1)}{2} = \frac{5 - 1}{2} = 2.$$
4. **Find c:**
$f'(x) = 2x$. Set $2c = 2 \Rightarrow c = 1$.

Final Answer: $c = 1.$

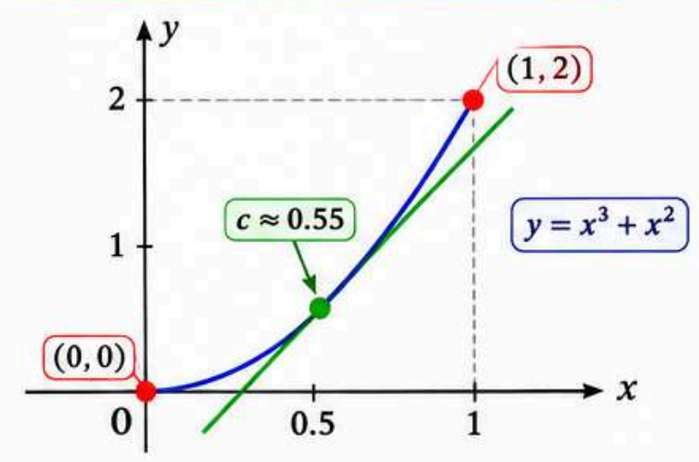


3 $f(x) = x^3 + x^2, \ [0, 1]$

Solution:

1. **Continuity:** f is continuous on $[0, 1]$ (polynomial).
2. **Differentiability:** f is differentiable on $(0, 1)$.
3. **Average rate of change:**
$$\frac{f(1) - f(0)}{1 - 0} = \frac{(1^3 + 1^2) - (0^3 + 0^2)}{1} = 2.$$
4. **Find c:**
$f'(x) = 3x^2 + 2x$. Set $3c^2 + 2c = 2 \Rightarrow 3c^2 + 2c - 2 = 0$.
$c = \frac{-1 \pm \sqrt{7}}{3}$. The value in $(0, 1)$ is $c = \frac{-1 + \sqrt{7}}{3} \approx 0.55$.

Final Answer: $c = \frac{-1 + \sqrt{7}}{3} \approx 0.55.$



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The Mean Value Theorem (MVT) – Exercises 4 – 6

Theorem Reminder:

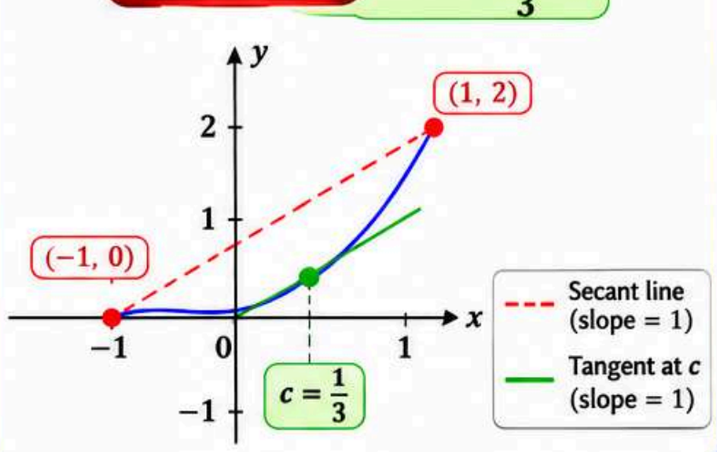
- **Mean Value Theorem (MVT):** If f is continuous on $[a, b]$ and differentiable on (a, b), then there exists $c \in (a, b)$ such that $f'(c) = \frac{f(b) - f(a)}{b - a}$.
- **Rolle's Theorem:** If f is continuous on $[a, b]$, differentiable on (a, b), and $f(a) = f(b)$, then there exists $c \in (a, b)$ such that $f'(c) = 0$.

4 $f(x) = x^3 + x^2$ on $[-1, 1]$

Solution:

1. **Continuity:** f is a polynomial, so it is continuous on $[-1, 1]$.
2. **Differentiability:** f is differentiable on $(-1, 1)$.
3. **Average slope:** $\frac{f(1) - f(-1)}{1 - (-1)} = \frac{2 - 0}{2} = 1.$
4. **Derivative:** $f'(x) = 3x^2 + 2x.$
5. **Find c:** Set $f'(c) = 1$:
$3c^2 + 2c = 1 \Rightarrow 3c^2 + 2c - 1 = 0$
$\Rightarrow (3c - 1)(c + 1) = 0$
$c = \frac{1}{3}$. or $c = -1$.
6. **Conclusion:** Reject $c = -1$ (endpoint), so $c = \frac{1}{3}$.

Final Answer: $c = \frac{1}{3}.$

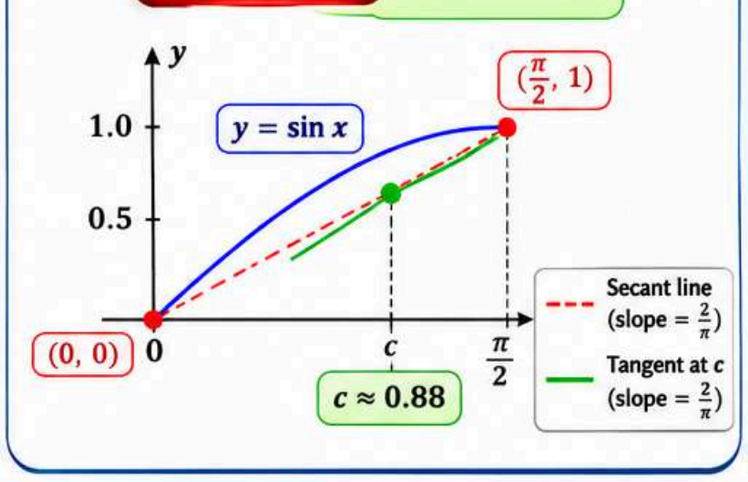


5 $f(x) = \sin x$ on $\left[0, \frac{\pi}{2}\right]$

Solution:

1. **Continuity:** f is continuous on $\left[0, \frac{\pi}{2}\right]$.
2. **Differentiability:** f is differentiable on $\left(0, \frac{\pi}{2}\right)$.
3. **Average slope:**
$$\frac{f\left(\frac{\pi}{2}\right) - f(0)}{\frac{\pi}{2} - 0} = \frac{\sin\left(\frac{\pi}{2}\right) - \sin(0)}{\frac{\pi}{2}} = \frac{1}{\frac{\pi}{2}} = \frac{2}{\pi}.$$
4. **Derivative:** $f'(x) = \cos x.$
5. **Find c:** Set $f'(c) = \frac{2}{\pi}$: $\cos c = \frac{2}{\pi}$.
$\Rightarrow c = \cos^{-1}\left(\frac{2}{\pi}\right) \approx 0.88$ radians.

Final Answer: $c \approx 0.88.$

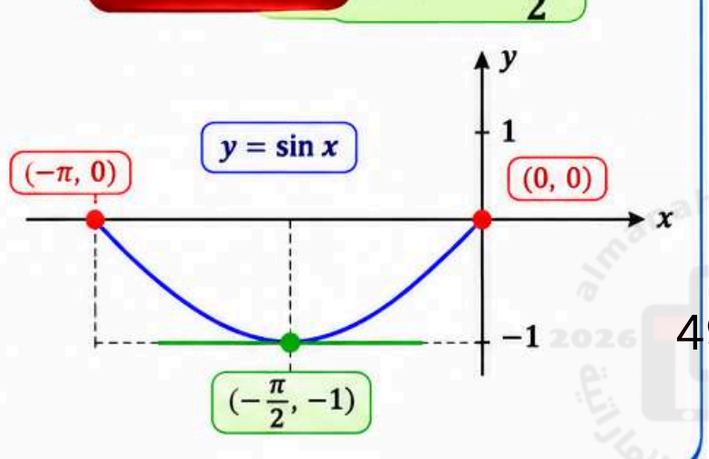


6 $f(x) = \sin x$ on $[-\pi, 0]$

Solution:

1. **Continuity:** f is continuous on $[-\pi, 0]$.
2. **Differentiability:** f is differentiable on $(-\pi, 0)$.
3. **Endpoint values:**
$f(-\pi) = \sin(-\pi) = 0$ and $f(0) = \sin 0 = 0$.
Thus $f(-\pi) = f(0)$.
4. **Apply Rolle's Theorem:**
Since f is continuous on $[-\pi, 0]$, differentiable on $(-\pi, 0)$, and $f(-\pi) = f(0)$, Rolle's Theorem applies (and consequently MVT), so $f'(c) = 0$ for some $c \in (-\pi, 0)$.
5. **Find c:** $f'(x) = \cos x$. Set $\cos c = 0$.
The value in $(-\pi, 0)$ is $c = -\frac{\pi}{2}$.

Final Answer: $c = -\frac{\pi}{2}$



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