حل تدريبات الدرس الثالث Maximum and Minimun Values من الوحدة الرابعة منهج ريفيل

Unit 4

Applications of Differentiation

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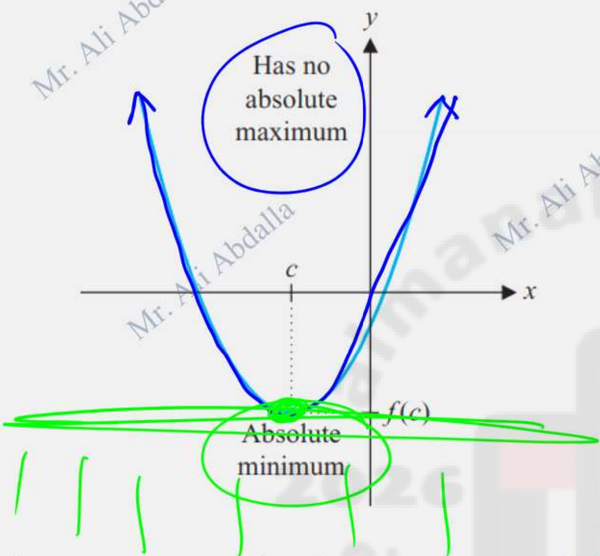
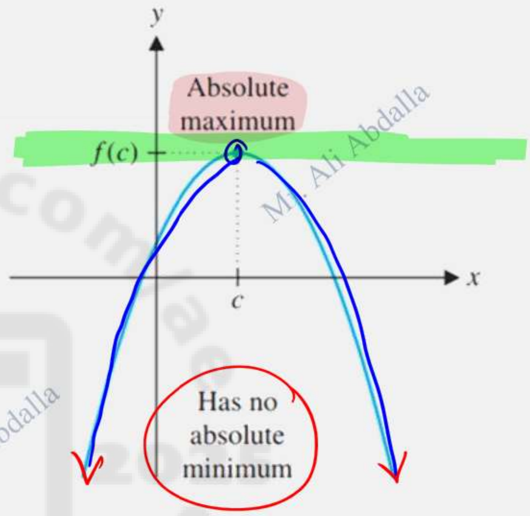
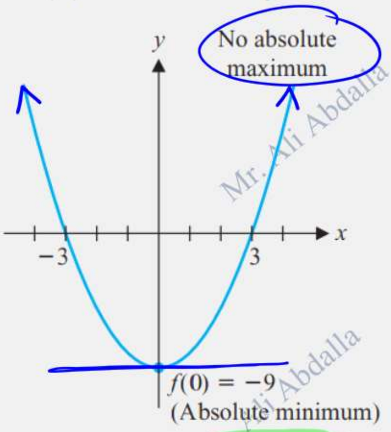
Definition 3.1

For a function f defined on a set S of real numbers and a number $c \in S$,
(i) $f(c)$ is the **absolute maximum** of f on S if $f(c) \geq f(x)$ for all $x \in S$ and
(ii) $f(c)$ is the **absolute minimum** of f on S if $f(c) \leq f(x)$ for all $x \in S$.
An absolute maximum or an absolute minimum is referred to as an **absolute extremum**. (The plural form of **extremum** is extrema.)



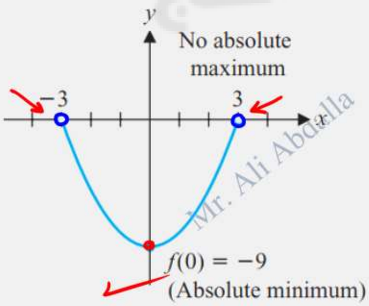


1) (a) Locate any absolute extrema of $f(x) = x^2 - 9$ on the interval $(-\infty, \infty)$.
(b) Locate any absolute extrema of $f(x) = x^2 - 9$ on the interval $(-3, 3)$.
(c) Locate any absolute extrema of $f(x) = x^2 - 9$ on the interval $[-3, 3]$.

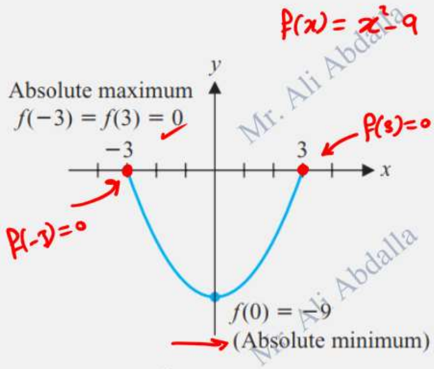


$y = x^2 - 9$ on $(-\infty, \infty)$

$f(0) = -9$

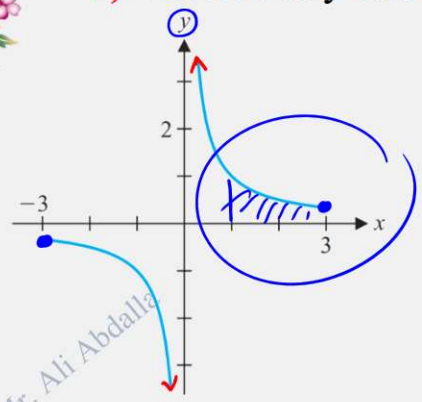


$y = x^2 - 9$ on $(-3, 3)$



$y = x^2 - 9$ on $[-3, 3]$

2) Locate any absolute extrema of $f(x) = \frac{1}{x}$, on $[-3,0) \cup (0,3]$



$y = 1/x$

$x = 0$ Vertical Asymptote

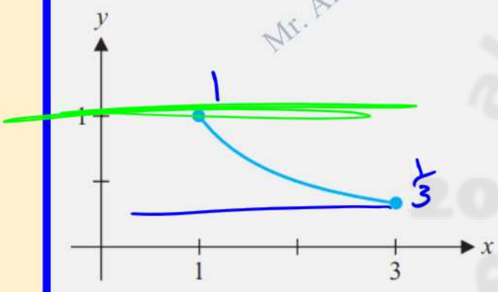
No Absolute Max

No Absolute Min

Theorem 3.1 (Extreme Value Theorem)

A continuous function f defined on a closed, bounded interval $[a, b]$ attains both an absolute maximum and an absolute minimum on that interval.

3) Locate any absolute extrema of $f(x) = \frac{1}{x}$, on $[1, 3]$

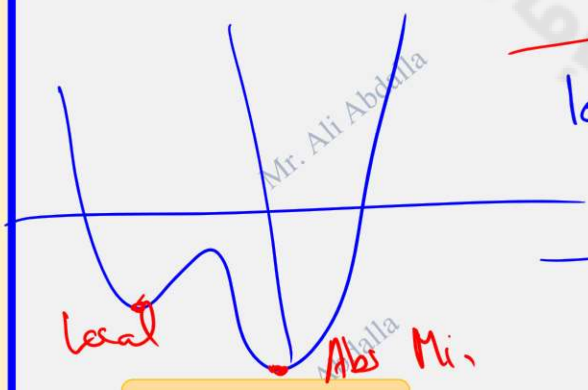


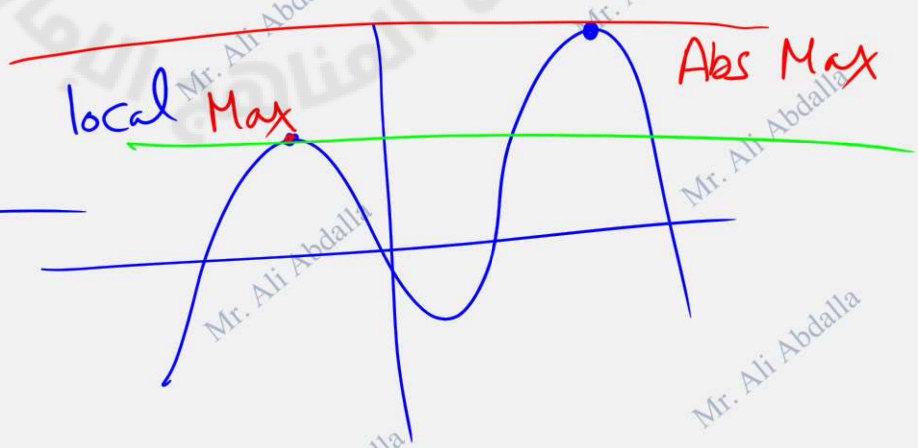
$y = 1/x$ on $[1, 3]$

$f(x) = \frac{1}{x}$

$x = 1$ $\quad f(1) = \frac{1}{1} = 1$ Abs. Max

$x = 3$ $\quad f(3) = \frac{1}{3}$ Abs Min





Definition 3.2

(i) $f(c)$ is the **local maximum** of f if $f(c) \geq f(x)$ for all x in some open interval containing c.

(ii) $f(c)$ is the **local minimum** of f if $f(c) \leq f(x)$ for all x for all x in some open interval containing c.

In either case, we call $f(c)$ a local extremum of f.

Notice

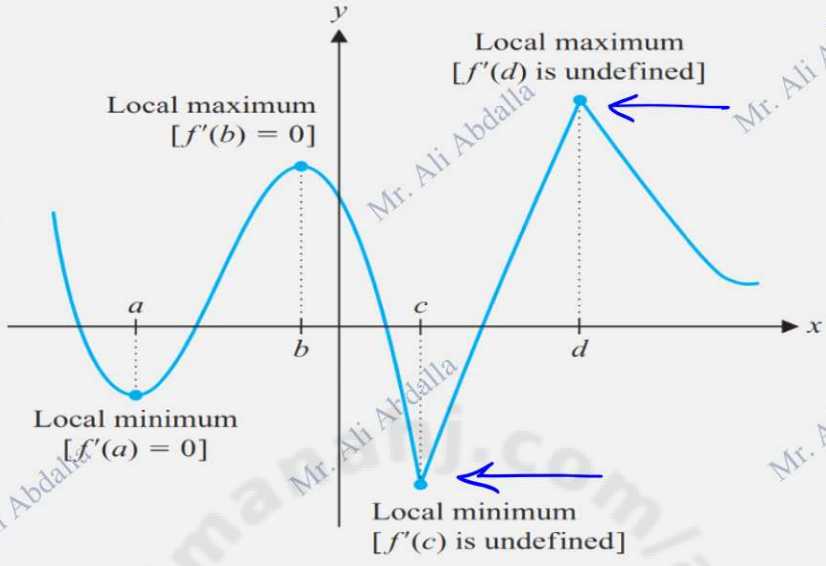
Max Min

Each local extremum seems to occur at

(A) a point where the tangent line is horizontal [i.e., where $f'(x) = 0$],

(B) ~~a point where the tangent line is vertical [where $f'(x)$ is undefined]~~

(C) a corner (cusp) [where $f'(x)$ is undefined].



Local extrema

Definition 3.3 critical number

A number c in the domain of a function f is called a **critical number** of f if $f'(c) = 0$ or $f'(c)$ is undefined.

$x = 1$
$x - 2$
$x = -3$

$[-4, 1]$

$f'(x) = 0$ $f'(x)$ undefined

Theorem 3.2 (Fermat's Theorem)

Suppose that $f(c)$ is a local extremum (local maximum or local minimum). Then c must be a critical number of f.

4) Find the critical numbers and local extrema of $f(x) = 2x^3 - 3x^2 - 12x + 5$ $(-\infty, \infty)$

$f'(x) = 0$

$f'(x)$ undefined

$f'(x) = 6x^2 - 6x - 12$

$f'(x) = 0 \Rightarrow 6x^2 - 6x - 12 = 0$

$x = 2, \; x = -1$

$f(2) = -15$ local Min

$f(1) = -8$ local Max

5) Find the critical numbers and local extrema of $f(x) = (3x+1)^{2/3}$.

$f'(x) = \frac{2}{\cancel{3}}(3x+1)^{-1/3} \cdot \cancel{3} = \frac{2}{\sqrt[3]{3x+1}}$

$f'(x) \neq 0$ $\quad$ $f'(x)$ undefined

$3x+1=0 \Rightarrow \boxed{x = -\frac{1}{3}}$

$f(x) = (3x+1)^{\frac{2}{3}} \leftarrow$ odd $\quad$ Domain $(-\infty, \infty)$ $\quad$ $-1 \quad -\frac{1}{3} \quad 0$

$x = -\frac{1}{3}$ critical number

$f(-\frac{1}{3}) = 0$ local Min

$f(-1) = 1.6$

$f(0) = 1$

$[f(x)]^{\frac{m}{n}} \leftarrow$ even $\quad f(x) \geq 0$

1.6 $\quad$ 0 $\quad$ 1

$-1 \quad -\frac{1}{3} \quad 0$

Theorem 3.3

Suppose that f is continuous on the closed interval $[a, b]$. Then, each absolute extremum of f must occur at an endpoint (a or b) or at a critical number.

REMARK 3.4

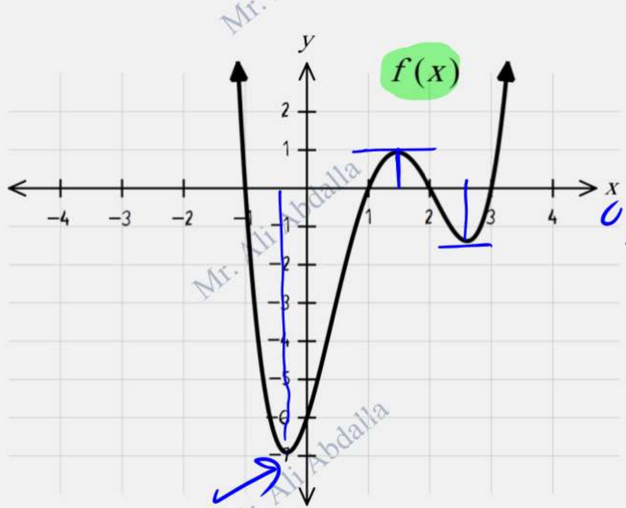
Theorem 3.3 gives us a simple procedure for finding the absolute extrema of a continuous function on a closed, bounded interval: $\quad x_1 \quad x_2 \quad x_3 \quad [a, b]$

1. Find all critical numbers in the interval and compute function values at these points.
2. Compute function values at the endpoints.
3. The largest of these function values is the absolute maximum and the smallest of these function values is the absolute minimum.

6) If $f(x)$ is defined on All Real Numbers, for each case find the critical numbers.

(a) If the following graph represent the function of $f(x)$.

$(-\infty, \infty)$



* $f(x)$ graph

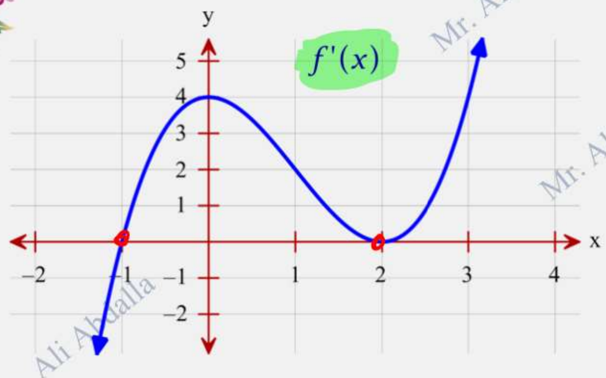
* قيم x المقابلة للقمم العظمى أو الصغرى

$x = -\frac{1}{2}$

$x = 1\frac{1}{2} = 3/2$

$x = 2.5 = 2\frac{1}{2} = 5/2$

7) If the following graph represent the first derivative of $f(x)$.



f'(x) graph

$f'(x) = 0$ → التقاطع مع محور x

$f'(x)$ undefined → Jump

$x = 2$

$x = -1$

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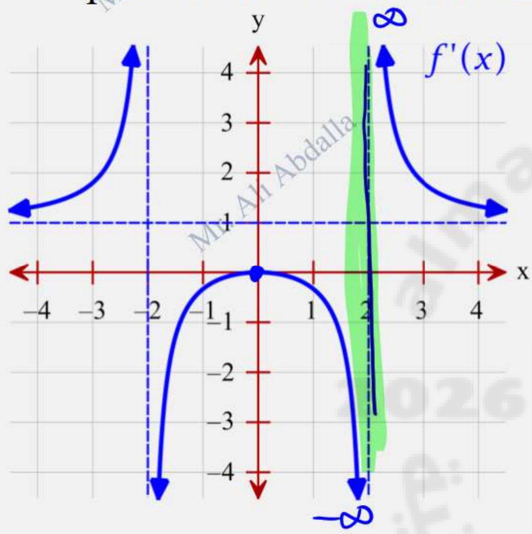
8) If $f(x)$ is defined and continuous on $(-\infty, \infty)$, and the following graph represent the first derivative of $f(x)$. Find the critical numbers.



* رسم $f'(x)$ نبحث عن محاور التقارب العمودية

✓ التقاطع مع محور x

$x = 2$, $x = -2$ V. Asy.

$x = 0$

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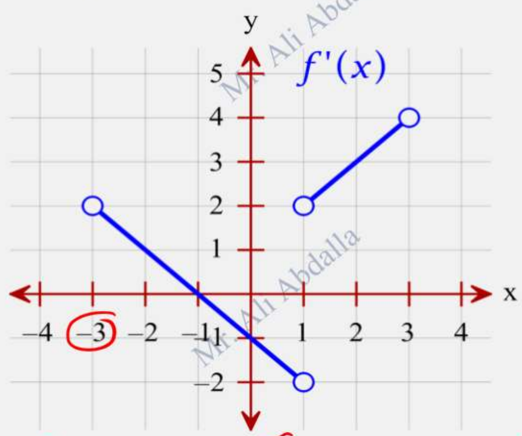
9) If $f(x)$ is defined and continuous on $(-3, 3]$, and the following graph represent the first derivative of $f(x)$. Find the critical numbers.



* في فترة محددة للمستمر $f'(x)$

1 ← بداية ونهاية الفترة اذا كانت موجودة من ضمن المجال

2 عدم الاتصال قفزة V.A

3 التقاطع مع محور x

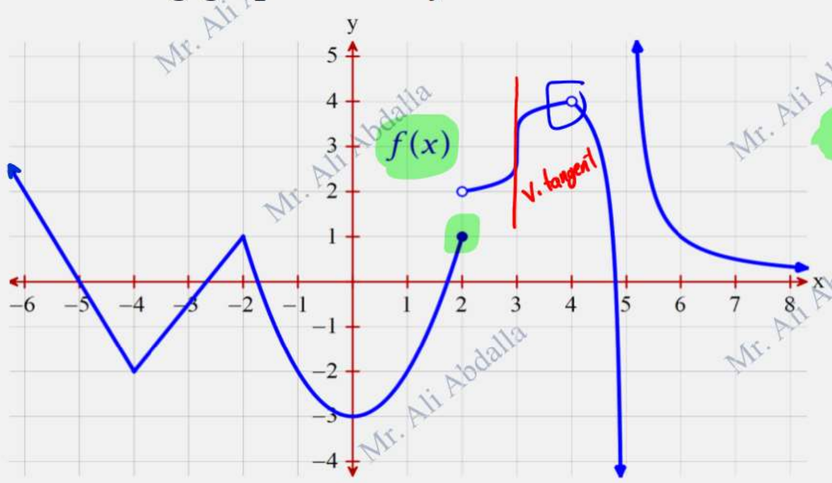
~~$x = -3$ خارج المجال~~

$x = 3$

$x = 1$ Jump $f'(x)$ undefined

$x = -1$ $f'(x) = 0$

10) By using the following graph find any critical number and write the reason.



$x=5$ V.A

ف، ع، ت، د

$x=4$ Hole

ف، ع، ت، د

Critical number	$x = -4$	$x = -2$	$x = 0$	$x = 2$	$x = 3$
reason	$f'(x)$ undefined corner	$f'(x)$ undefined cusp	$f'(0)=0$ Horizontal tangent Min	Jump $f'(x)$ undefined	Vertical tangent $f'(x)$ undefined

At $x = 5$ and $x = 4$ No critical numbers, why?

V. Asym   Hole   not included in the domain

===

11) Find the critical numbers of

A) $f(x) = xe^{-x^2}$ $(-\infty, \infty)$

B) $f(x) = e^{-x^2}$ $x=0$

$$f'(x) = 1 \cdot e^{-x^2} + x \cdot (-2x e^{-x^2})$$

$$= e^{-x^2}\left[1-2x^2\right] = \frac{1-2x^2}{e^{x^2}}$$

$f'(x) = 0$ $f'(x) =$ undefined

$1-2x^2=0$

$2x^2=1$

$x^2=\frac{1}{2}$

$x = \pm\frac{1}{\sqrt{2}} = \frac{\pm\sqrt{2}}{2}$

12) Find the critical numbers and local extrema of $f(x) = \sin x \cos x$ in the interval $[0, 2\pi]$

$f'(x) = \cos x \cdot \cos x + \sin x(-\sin x)$

$= \cos^2 x - \sin^2 x$

$f'(x)$ undefined

$f'(x) = 0$

$\cos^2 x - \sin^2 x = 0 \Rightarrow \sin^2 x = \cos^2 x \quad \div \cos^2 x$

$\pi-\theta$ S | A θ

$\pi+\theta$ T | C $2\pi-\theta$

$\boxed{\tan^2 x = 1} \Rightarrow \tan x = \pm 1$

S | A

T | C

$\tan x = 1$ ⊕

$x = \frac{\pi}{4}$ I: $\frac{\pi}{4}$; III: $\pi + \frac{\pi}{4} = \frac{5\pi}{4}$

$\tan x = -1$ ⊖

$x = \frac{\pi}{4}$ II: $\pi - \frac{\pi}{4} = \frac{3\pi}{4}$; IV: $2\pi - \frac{\pi}{4} = \frac{7\pi}{4}$

critical number

$\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

$f(x) = \sin x \cos x$

$f(0)$ 0 $f(2\pi)$ 0 $f(\frac{\pi}{4})$ $\frac{1}{2}$ $f(\frac{3\pi}{4})$ $-\frac{1}{2}$ $f(\frac{5\pi}{4})$ $\frac{1}{2}$ $f(\frac{7\pi}{4})$ $-\frac{1}{2}$

13) Find the critical numbers and absolute extrema of $f(x) = x^3 - 3x + 1$ on

a) $[0, 2]$ b) $[-3, 2]$

$f'(x) = 3x^2 - 3$

$f'(x) = 0$

$3x^2 - 3 = 0$

$3x^2 = 3$

$\boxed{x = \pm 1}$

$\boxed{a}$ $[0, 2]$

$x = 1 \in [0, 2]$

$x = -1 \notin [0, 2]$

critical number

$\boxed{x = 1}$

(b) $[-3, 2]$

$x = 1 \in [-3, 2]$

$x = -1 \in [-3, 2]$

critical

$x = -1$ ✓

$x = 1$ ✓

14) Find the critical numbers and absolute extrema of $f(x) = x^2e^{-4x}$ on

a) $[-2, 0]$ b) $[0, 4]$

15) If $f(x) = 2x^3 + ax^2 + bx$ has critical point at $x = 1$ and $x = -2$ find the value of a and b.

$f'(x) = 6x^2 + 2ax + b$

$f'(1) = 0$ $6(1)^2 + 2a(1) + b = 0$ $\boxed{2a + b = -6}$

$f'(-2) = 0$ $6(-2)^2 + 2a(-2) + b = 0$ $\boxed{-4a + b = -24}$

$6a = 18$

$\boxed{a = 3}$

$2(3) + b = -6$

$\boxed{b = -12}$

For questions (16 – 22) Find the critical numbers for each function.

16) $f(x) = \begin{cases} 7 - 2x^2, & x \leq 1 \\ x^2 - 12x, & x > 1 \end{cases}$

$x \leq 1 \qquad x > 1$

$f: \quad 7 - 2x^2 \quad | \quad x^2 - 12x$

$f': \quad -4x \quad | \quad 2x - 12$

1

$-4x = 0 \qquad 2x - 12 = 0$

$\boxed{x = 0} \qquad \boxed{x = 6}$

$-4 \qquad 2(1) - 12 = -10$

$x = 1 \quad f'(1)$ undefined

Critical number

$x = 1, x = 0, x = 6$

17) $f(x) = \dfrac{x}{\sqrt{x^2+1}}$ Domain $(-\infty, \infty)$

$$f'(x) = \frac{1 \cdot \sqrt{x^2+1} - x \cdot \left(\frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+1}}\right)}{\left(\sqrt{x^2+1}\right)^2} \qquad \frac{\cdot \sqrt{x^2+1}}{\sqrt{x^2+1}}$$

$$= \frac{(x^2+1) - x^2}{\left(\sqrt{x^2+1}\right)^3} = \frac{1}{(x^2+1)^{3/2}}$$

$f'(x) \neq 0 \qquad f'(x)$ undefined $\quad x^2 + 1 \neq 0$

No critical number

واجب

18) $f(x) = \sqrt{3}\sin x + \cos x$

19) $f(x) = x^{3/4} - 4x^{1/4}$

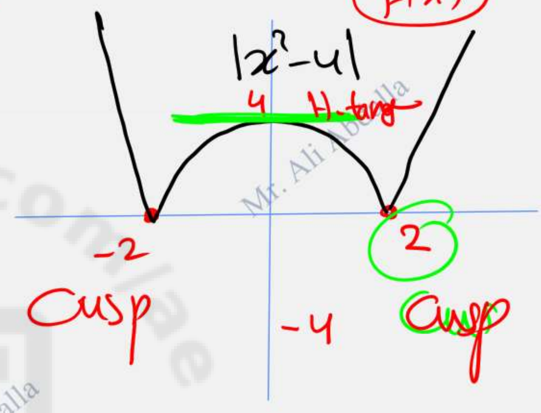
20) $f(x) = |x^2 - 4|$

$g(x) = x^2 - 4$

$x^2 - 4 = 0$

$f(x)$

$x = 2$
$x = -2$
$x = 0$



21) $f(x) = \sqrt[3]{x^3 - 3x^2}$

22) $f(x) = \begin{cases} \sin x & , -\pi < x < \pi \\ -\tan x & , |x| \geq \pi \end{cases}$

-tan | sin x | -tan x

$-\pi$ π

$x \geq \pi$ or $x \leq -\pi$

23) Find the critical numbers and local extrema of $f(x) = \sin x + \cos x$ in the interval $[0, 2\pi]$ and $\left[\frac{\pi}{2},\ \pi\right]$

24) Find the critical numbers and local extrema of $f(x) = \tan^{-1} x^2$ in the interval $[0, 1]$ and $[-3,\ 4]$

24) Find the critical numbers and local extrema of $f(x) = \frac{3x}{x^2+16}$ in the interval $[0,2]$ and $[0,\ 6]$

25) Find the critical numbers and local extrema of $f(x) = 2x\sqrt{x+1}$

Notes: on $f(x)$ Domain

1- To find critical numbers of a function $f(x)$

- ❑ find the value(s) of x where $f'(x) = 0$ and $f'(x)$ is undefined.

لإيجاد الأعداد الحرجة للدالة $f(x)$

أوجد قيم x التي تكون عندها قيمة المُشتقة مساوية للصفر أو المُشتقة غير معرَّفة حسب الحالات التي سبق ذكرها.

2- To find critical numbers from the graph of the function $f(x)$

- ❑ find the value(s) of x where $f(x)$ has maximum or minimum values.

لإيجاد الأعداد الحرجة من الرسم البياني للدالة $f(x)$: نوجد قيم x التي يكون للدالة عندها قيم عظمى أو صغرى

3- To find critical numbers from the graph of the function $f'(x)$

- ❑ find the value(s) of x where $f'(x)$ intersect with x-axis.

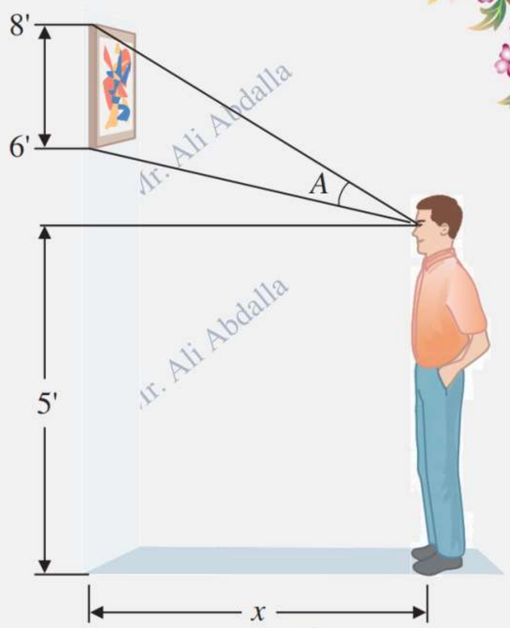
لإيجاد الأعداد الحرجة من الرسم البياني للدالة $f'(x)$: نوجد قيم x التي يتقاطع منحنى الدالة $f'(x)$ مع محور x

في المجال المُعَرَّف عليه الدالة

Suppose a painting hangs on a wall as in the figure. The frame extends from 6 feet to 8 feet above the floor. A person whose eyes are 5 feet above the ground stands x feet from the wall and views the painting, with a viewing angle A formed by the ray from the person's eye to the top of the frame and the ray from the person's eye to the bottom of the bottom of the frame.

A) Find the value of x that maximizes the viewing angle A.

B) What changes if the person's eyes are 6 feet above the ground?



You are sitting in a classroom next to the wall looking at the blackboard at the front of the room. The blackboard is 12 ft long and starts 3 ft from the wall you are sitting next to. Show that your viewing angle is

$$\alpha = \cot^{-1}\left(\frac{x}{15}\right) - \cot^{-1}\left(\frac{x}{3}\right)$$

if you are x ft from the front wall.